# Realistic Roofs over a Rectilinear Polygon Revisited*

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**Abstract.** A common task in automatically reconstructing a three dimensional city model from its two dimensional map is to compute all the possible roofs over the ground plans. A *roof* over a simple polygon in the $xy$-plane is a terrain over the polygon such that each face $f$ of the terrain is supported by a plane passing through at least one polygon edge and making a dihedral angle $\frac{\pi}{4}$ with the $xy$-plane [3]. This definition, however, allows roofs with faces isolated from the boundary of the polygon and local minimum edges inducing pools of rainwater. Recently, Ahn et al. [1,2] introduced "realistic roofs" over a simple rectilinear polygon $P$ with $n$ vertices by imposing two additional constraints under which no isolated faces and no local minimum vertices are allowed. Their definition is, however, too restrictive that it excludes a large number of roofs with no local minimum edges. In this paper, we propose a new definition of realistic roofs corresponding to the class of roofs without isolated faces and local minimum edges. We investigate the geometric and combinatorial properties of realistic roofs and show that the maximum possible number of distinct realistic roofs over $P$ is at most $1.3211^m \binom{m}{\lfloor \frac{m}{2} \rfloor}$, where $m = \frac{n-4}{2}$. We also present an algorithm that generates all combinatorial representations of realistic roofs.

## 1 Introduction

A common task in automatically reconstructing a three dimensional city model from its two dimensional map is to compute all the possible roofs over the ground plans of its buildings extensively [5,6,9,7,8,10]. For some applications, a correct or reasonable roof over a building is chosen from the candidates using some other information such as its images.

Aichholzer et al. [3] defined a *roof* over a simple (not necessarily rectilinear) polygon in the $xy$-plane as a terrain over the polygon such that each face of the terrain is supported by a plane passing through at least one polygon edge and making a dihedral angle $\frac{\pi}{4}$ with the $xy$-plane. This definition, however, is

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not tight enough that it allows roofs with faces isolated from the boundary of the polygon (Figure 1(a)) and local minimum edges (Figure 1(b)) which are undesirable for some practical reasons – for example, a local minimum edge serves as a pool of rainwater. Note that a pool of rainwater on a roof always contains a local minimum edge.

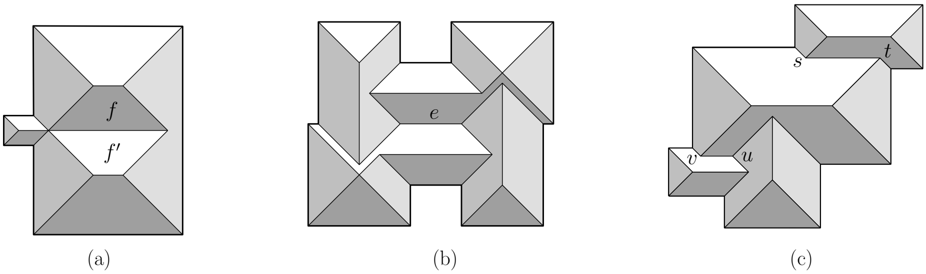


**Fig. 1.** (a) A roof with isolated faces $f$ and $f'$. (b) A roof with a local minimum edge $e$. (c) Not a realistic roof according to Definition 1; vertex $u$ has no adjacent vertex $v$ that is lower than $u$.

### 1.1 Related Work

Brenner [6] designed an algorithm that computes all the possible roofs over a rectilinear polygon, but no polynomial bound on its running time is known. Recently, Ahn et al. [1,2] introduced "realistic roofs" over a rectilinear polygon $P$ with $n$ vertices by imposing two additional constraints as follows.

**Definition 1 ([1]).** *A realistic roof over a simple rectilinear polygon $P$ is a roof over $P$ satisfying the following constraints.*
*C1. Each face of the roof is incident to at least one edge of $P$.*
*C2. Each vertex $u$ of the roof is higher than at least one of its neighboring vertices.*

They showed some geometric and combinatorial properties of realistic roofs, including a connection to the *straight skeleton* [3,4]. Consider a roof $R^*(P)$ over $P$ constructed by *shrinking process*, where all of the edges of $P$ move inside, being parallel to themselves, with the same speed while moving upward along the $z$-axis with the same speed. Ahn et al. [1,2] showed that $R^*(P)$ is unique, its projection on the $xy$-plane is the straight skeleton of $P$, and it is the point-wise highest realistic roof over $P$. From the fact that $R^*(P)$ does not have a "valley", we can construct another realistic roof $R$ over $P$ which is different to $R^*(P)$ by adding a set of "compatible valleys" to $R^*(P)$. They showed that the number of realistic roofs lies between 1 and $\binom{m}{\lfloor \frac{m}{2} \rfloor}$ where $m = \frac{n-4}{2}$, and presented an output sensitive algorithm generating all combinatorial representations of realistic roofs over $P$ in $O(1)$ time per roof, after $O(n^4)$ preprocessing time.

### 1.2 Our Results

Constraint *C1* in Definition 1 is to exclude roofs with isolated faces and constraint *C2* is introduced to avoid pools of rainwater. However, *C2* is too restrictive that it also excludes a large number of roofs with no local minimum edges. For example, the roof in Figure 1 (c) is not realistic according to Definition 1 – $u$ is a local minimum vertex – though rainwater can be drained well along $uv$. Therefore, Definition 1 by Ahn et al. does describe only a proper subset of "realistic" roofs.

We introduce a new definition of realistic roofs by replacing *C2* of Definition 1 with a relaxed one that excludes roofs with local minimum edges.

**Definition 2.** *A realistic roof over a simple rectilinear polygon $P$ is a roof over $P$ satisfying the following constraints.*
*C1. Each face of the roof is incident to at least one edge of $P$.*
*C2′. For each roof edge $uv$, $u$ or $v$ is higher than at least one of its neighboring vertices.*

From now on, we use Definition 2 for realistic roofs unless stated explicitly. One important difference is that our realistic roofs do not have local minimum edges. Our definition corresponds to the class of roofs without isolated faces and local minimum edges exactly. This is often required for roofs in the real-world, as rainwater cannot be well drained from a local minimum along the roof surface.

Our main results are threefold:

1. We provide a new definition of "realistic roofs" corresponding to the real-world roofs and investigate geometric properties of them.
2. We show that the maximum possible number of realistic roofs over a rectilinear $n$-gon is at most $1.3211^m\binom{m}{\lfloor \frac{m}{2} \rfloor}$, where $m = \frac{n-4}{2}$.
3. We present an algorithm that generates all combinatorial representations of realistic roofs over a rectilinear $n$-gon, including roofs with *open valleys* only and roofs with *half-open valleys*. Precisely, it generates a roof with *open valleys* only in $O(1)$ time after $O(n^4)$ preprocessing time [1]. For each such roof, it generates $O(1.3211^m)$ roofs with half-open valleys in time $O(m1.3211^m)$.

## 2 Preliminary

For a point $p$ in $\mathbb{R}^3$, we denote by $z(p)$ the $z$-coordinate of $p$ and by $\overline{p}$ the orthogonal projection of $p$ onto the $xy$-plane. For a point $p \in R$, let $D(p)$ be a square centered at $\overline{p}$ with side length $z(p)$. For any two points $s, t \in \mathbb{R}^2$, let $d_\infty(s,t)$ be $L_\infty$ distance between $s$ and $t$, and $d_\infty(s,A) := \inf_{a \in A} d_\infty(s,a)$ for any set $A$ in $\mathbb{R}^2$. We denote by $\partial P$ the boundary of $P$ and by edge$(f)$ the edge of $\partial P$ incident to a face $f$ of a roof.

**Lemma 1.** *([1]) Let $R$ be a roof over a rectilinear simple polygon $P$. The followings hold.*

1. *For any point $p \in R$, we have $z(p) \le d_\infty(\overline{p}, \partial P)$ and $D(p) \subseteq P$.*
2. *For each edge $e$ of $P$, there exists a unique face $f$ of $R$ incident to $e$.*
3. *Every face $f$ of $R$ is monotone with respect to the line containing* edge$(f)$*.*

***Edge Types.*** Edges of a realistic roof $R$ over $P$ can be classified into two groups, *convex edges* and *reflex edges*. An edge $e$ of a roof $R$ is called *convex* if $R$ is locally convex along $e$, and an edge $e'$ is called *reflex* if $R$ is locally reflex along $e'$. Also, we will call an edge $e$ a *valley* if $e$ is reflex and parallel to the $xy$-plane, and call an edge $e'$ a *ridge* if $e'$ is convex and parallel to the $xy$-plane.

## 3 Valleys of a Realistic Roof

In this section, we investigate local structures of realistic roofs. Ahn et al. showed that realistic roofs with Definition 1 can have vertices of 5 different types for a ridge and vertices which are not incident to a valley or a ridge are degenerated forms of valleys or ridges. Since replacing constraint *C2* to *C2*$'$ does not affect ridges, so we care about only valleys.

We define three types of valleys and describe the structures of valleys that a realistic roof can have. We call a vertex of a roof *open* if it is higher than at least one of its neighboring vertices, and *closed* otherwise. We call a valley *open* if both corners are open vertices, *half-open* if one corner is an open vertex and the other is a closed vertex, and *closed* if both corners are closed vertices. By Definition 2, a realistic roof can contain open and half-open valleys unless they make an isolated face, but it does not contain closed valleys. Ahn et al. showed that each open valley always has the same structure as $st$ in Figure 1(c). More specifically, they showed that there are only 5 possible configurations at a corner of a valley of a roof because of the roof constraints such as the monotonicity of a roof, and the slope and orientations of faces as illustrated in Figure 2. They showed that a realistic roof has an open valley both corners of which are of type (v1) and oriented symmetrically along the valley. Also each corner of an open valley is connected to a reflex vertex of $P$ by a reflex edge. We call these two reflex vertices a *candidate pair*.

In the following we investigate the structure of a half-open valley that a realistic roof can have. It is not difficult to see that the open corner is always of type

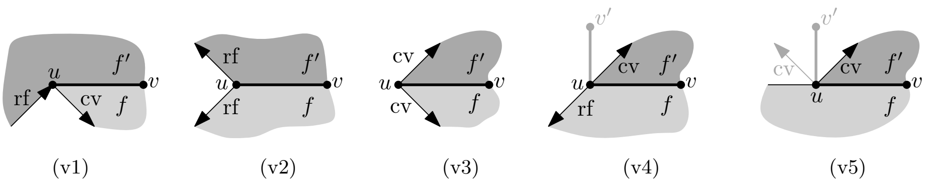


**Fig. 2.** 5 configurations around a vertex of a valley $uv$, where rf denotes a reflex edge and cv denotes a convex edge. Each convex or reflex edge is oriented from the endpoint with smaller $z$-coordinate to the other one with larger $z$-coordinate.

(v1) – all the others cannot have a lower neighboring vertex because of the roof constraints, that is, they are all closed. We will show that the closed corner of a realistic roof is always of type (v2). For this, we need a few technical lemmas. A proof of the following lemma can be found in the full version of the paper.

**Lemma 2.** *Let $uv$ be a valley and $uv'$ be a convex edge connected to $u$. Also, let $\ell$ be the line in the $xy$-plane passing through $\overline{v}$ and orthogonal to $\overline{uv}$. Then the face $f$ incident to both $uv$ and $uv'$ has* edge$(f)$ *in the half-plane of $\ell$ in the $xy$-plane not containing $\overline{u}$.*

Imagine that a face $f$ is incident to a valley $uv$ and two convex edges one of which is incident to $u$ and the other to $v$. Then both convex edges must lie in the same side of the plane containing $uv$ and parallel to the $z$-axis. Since both convex edges make an angle 45° with $uv$ in their projection on the $xy$-plane, $f$ cannot have a ground edge, by Lemma 2, that is, $f$ is *isolated*.

**Lemma 3.** *Let $uv$ be a half-open valley of a realistic roof where $u$ is closed and $v$ is open. Then $v$ is of type* (v1) *and $u$ is of type* (v2).

*Proof.* If $u$ is of type (v3), then one of two faces incident to $uv$ becomes isolated by Lemma 2. If $u$ is of type (v5), then there always is another valley $uv'$ that is orthogonal to $uv$ and has a closed corner at $u$ of type (v3) as shown in Figure 2. Therefore one of faces incident to $uv'$ is isolated.

Next, assume that $u$ is of type (v4). Then there always is another valley $uv'$ orthogonal to $uv$. Therefore, we need to check two connected valleys $uv$ and $uv'$ simultaneously. Figure 3 illustrates all possible combinations of these two valleys. For cases (a) and (b), there is an isolated face incident to $uv$ or $uv'$. For case (c), let $f$ and $f'$ be the faces incident to $uv$ and $uv'$ respectively, as shown in Figure 3(c). Let $\ell$ ($\ell'$ resp.) be a line in the $xy$-plane which contains $\overline{uv}$ ($\overline{uv'}$ resp.). By Lemma 2, edge$(f)$ and edge$(f')$ are located in opposite quadrants defined by $\ell$ and $\ell'$. Then one of $f$ and $f'$ violate the monotonicity property (3) of Lemma 1. All valleys containing a vertex of type (v4) are invalid; the remaining closed corner is of type (v2). ◻

Now we are ready to describe the structure of a half-open valley. A proof of the following lemma can be found in the full version of the paper.

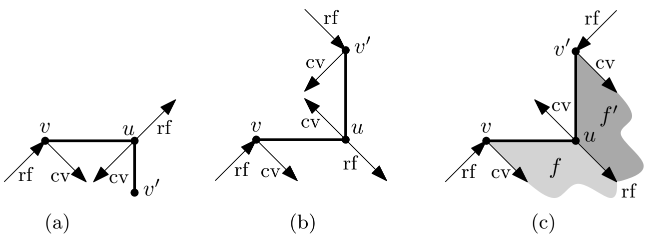


**Fig. 3.** Three possible combinations around a (v4) type vertex

**Lemma 4.** *Let uv be a half-open valley where u is closed and v is open. Then uv is associated with three reflex vertices of P that have mutually different orientations as in Figure 4.*

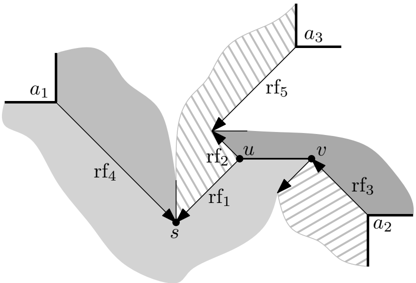


**Fig. 4.** A half-open valley $uv$ must be connected to three reflex vertices $a_1, a_2$ and $a_3$ via 5 reflex edges. Call the vertex $s$ which is incident to $\mathrm{rf}_1$ and $\mathrm{rf}_4$ as a *peak point* of $uv$.

## 4 Realistic Roofs with Half-Open Valleys

***Three Reflex Vertices from a Half-Open Valley.*** We investigate local and global properties of half-open valleys. From Lemma 4, we know that a half-open valley is associated with three reflex vertices that have mutually different orientations. Therefore, we need a condition to check whether chosen three reflex vertices could *induce* a half-open valley. Let $a_1$, $a_2$ and $a_3$ be reflex vertices which have mutually different orientations. Without loss of generality, we can assume that these vertices are arranged as in Figure 4. Let the horizontal difference of two vertices $a_i$ and $a_j$ be $d_h(a_i, a_j)$ and the vertical difference be $d_v(a_i, a_j)$. We now consider two squares and one rectangle to determine whether these three vertices form a half-open valley. Let $r_1$ be the square with $a_1$ on its top left corner and side length $d_h(a_1, a_3)$. Let $r_2$ be the rectangle with $a_2$ on its bottom right corner with height $d_v(a_1, a_2)$ and width $d_v(a_1, a_2) + d_h(a_2, a_3)$. Finally, let $r_3$ be the square with $a_3$ on its top right corner and side length $d_v(a_2, a_3)$.

Next, define three rectilinear subpolygons of $P$. Let $P_1$ be the polygon formed by $r_1 \cup r_2$ and the cut off portion of $P$ below $r_1$, $r_2$ and $r_3$ (Figure 5(a)). Let $P_2$ be the polygon formed by $r_2 \cup r_3$ and the cut off portion of $P$ right of $r_1$, $r_2$ and $r_3$ (Figure 5(b)). Let $P_3$ be the polygon formed by $r_1 \cup r_3$ and the cut off portion of $P$ above of $r_1$, $r_2$ and $r_3$ (Figure 5(c)). We now present the following lemma, and a proof of the lemma can be found in the full version of the paper.

**Lemma 5.** *Three reflex vertices $a_1, a_2$ and $a_3$ form a half-open valley $uv$ if and only if $(r_1 - a_1) \cap \partial P = \emptyset$, $(r_2 - a_2) \cap \partial P = \emptyset$ and $(r_3 - a_3) \cap \partial P = \emptyset$.*

If three reflex vertices $a_1, a_2$ and $a_3$ induce a half-open valley $uv$, we call the triple $(a_1, a_2, a_3)$ a *candidate triple* of $uv$. Assume that three reflex vertices of a candidate triple are ordered as depicted in Figure 4, or the mirror image of Figure 4. Also, We will call $r_1 \cup r_2 \cup r_3$ as the *free space* of $uv$.

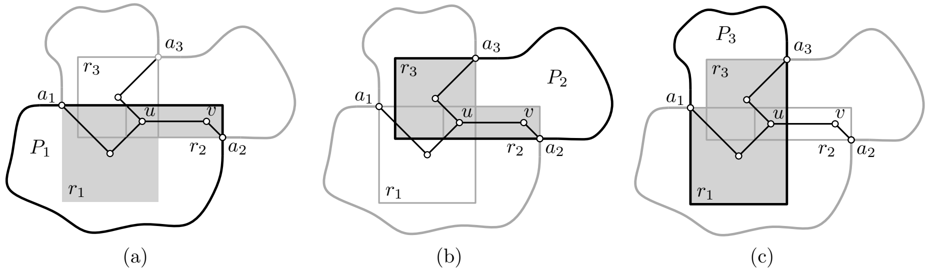


**Fig. 5.** Dividing $P$ into three rectilinear subpolygons, $P_1, P_2$ and $P_3$, *along* a half-open valley $uv$

***Compatibility.*** What we have to do next is to consider compatibilities. There are two cases: compatibility between two half-open valleys, and compatibility between an open valley and a half-open valley. We start with a lemma which states the compatibility between two open valleys.

**Lemma 6 ([1]).** *Let* $(a_1, a_2)$ *and* $(a_1', a_2')$ *be the candidate pairs for open valleys* $uv$ *and* $u'v'$*, respectively.* $(a_1, a_2)$ *and* $(a_1', a_2')$ *are compatible if and only if* $\overline{C}_{a_1a_2} \cap \overline{C}_{a_1'a_2'} = \emptyset$*, where* $\overline{C}_{a_1a_2} := a_1\overline{u} \cup \overline{uv} \cup \overline{v}a_2$ *and* $\overline{C}_{a_1'a_2'} := a_1'\overline{u'} \cup \overline{u'v'} \cup \overline{v}a_2'$.

Following two lemmas describe the compatibilities between two half-open valleys, and between a half-open valley and an open valley.

**Lemma 7.** *Let* $(a_1, a_2, a_3)$ *and* $(a_1', a_2', a_3')$ *be candidate triples for two half-open valleys* $uv$ *and* $u'v'$*. Then* $uv$ *and* $u'v'$ *are compatible if*

1. *All* $a_1, a_2$ *and* $a_3$ *are contained in one of* $\partial P \backslash \{a_1', a_2', a_3'\}$.
2. *Let* $P_{1a'}, P_{2a'}$ *and* $P_{3a'}$ *be rectilinear subpolygons of* $P$ *divided by* $u'v'$ *as we did before. If condition 1 is hold, then the free space of* $uv$ *is contained in one of* $P_{1a'}, P_{2a'}$ *and* $P_{3a'}$ *that contains* $a_1, a_2$ *and* $a_3$ *in its boundary.*

*Proof.* Lemma 4 shows that a half-open valley must be connected to three reflex vertices via 5 reflex edges. Suppose that condition 1 does not hold. It makes some of reflex edges of $uv$ and $u'v'$ crossing in their projection on the $xy$-plane. Among crossing edges, the lower edge cannot appear as an edge of a realistic roof. Hence, condition 1 is required.

Figure 6(a) shows the only possible configuration which violates condition 2 while satisfying condition 1. Consider two peak points of $uv$ and $u'v'$, $s$ and $s'$. We cannot get a proper face between $s$ and $s'$ because of their heights. Therefore we cannot get a realistic roof which contains $uv$ and $u'v'$ in this case.

Suppose that $uv$ and $u'v'$ satisfy both conditions 1 and 2. First, divide $P$ into three pieces, $P_{1a}, P_{2a}$ and $P_{3a}$ along $uv$. Next, divide one of $P_{1a}, P_{2a}$ and $P_{3a}$ which contains $a_1', a_2'$ and $a_3'$, into three pieces along $u'v'$. The latter division is possible because of condition 2. After two divisions, we get 5 rectilinear subpolygons of $P$, denoted by $P_1, \ldots, P_5$. By taking the upper envelope of the roofs $R^*(P_i)$, we can get a realistic roof which contains $uv$ and $u'v'$. ⊔⊓

**Lemma 8.** *Let $(a_1, a_2, a_3)$ be a candidate triple for a half-open valley $uv$ and $(a_1', a_2')$ be a candidate pair for an open valley $u'v'$. $uv$ and $u'v'$ are compatible if*

1. *Both $a_1'$ and $a_2'$ are contained in one of $\partial P \backslash \{a_1, a_2, a_3\}$.*
2. *Let $P_{1a}, P_{2a}$, and $P_{3a}$ be rectilinear subpolygon of $P$ divided by $uv$, as we did before. If condition 1 is hold, then $a_1'$ and $a_2'$ are contained in one of $P_{1a}, P_{2a}$, and $P_{3a}$, denoted by $P_a$. If $P_{1a} = P_a$, then $r_1$ of $uv$, denoted by $r_{1a}$, and the smallest axis parallel rectangle containing $a$ and $a'$, denoted by $B_{a_1'a_2'}$, satisfy the pseudo-disk property: $r_{1a}$ and $B_{a_1'a_2'}$ do not cross.*

*Proof.* Lemma 4 shows that a half-open valley must be connected to three reflex vertices via 5 reflex edges. Also an open valley must be connected to two reflex vertices via two reflex edges. As we see in the proof of Lemma 7, violating condition 1 makes some reflex edges of $uv$ and $u'v'$ crossing in their projection on the $xy$-plane, so condition 1 is required.

Suppose that $r_{1a}$ and $B_{a_1'a_2'}$ violate the pseudo-disk property. From the height difference between the peak point of $uv$ and the open valley $u'v'$, we can not get a proper face between them (Figure 6(b)). Therefore we cannot get a realistic roof which contains $uv$ and $u'v'$.

Ahn et al. showed how to construct a realistic roof $R$ over $P$ with a candidate pair $(a_1', a_2')$ for an open valley $u'v'$ : Divide $P$ into two pieces along $\overline{C}_{a_1'a_2'}$; Make two rectilinear subpolygon $P_1'$ and $P_2'$ by attaching $B_{a_1'a_2'}$ to each divided part; Take the upper envelope of $R^*(P_1') \cup R^*(P_2')$.

Suppose that $uv$ and $u'v'$ satisfy both condition 1 and 2. First, divide $P$ into three pieces, $P_{1a}, P_{2a}$, and $P_{3a}$. Next, divide $P_a$ which contains $a_1'$ and $a_2'$ into two pieces by using $u'v'$. After that, we can get 4 rectilinear subpolygons of $P$, denoted by $P_1, \ldots, P_4$. By taking the upper envelope of the roofs $R^*(P_i)$, we can get a realistic roof which contains $uv$ and $u'v'$. ∎

Let $V$ be a set of candidate pairs and candidate triples. If any two elements of $V$ satisfy Lemma 6 or Lemma 7 or Lemma 8, we can find a unique realistic roof $R$ whose valleys correspond to $V$. Also, we call such $V$ as a compatible set of $P$. Now we can conclude the following theorem.

**Theorem 1.** *Let $P$ be a rectilinear polygon with $n$ vertices and $V$ be a compatible set of $k$ candidate pairs and $l$ candidate triples with respect to $P$. Then there exists a unique realistic roof $R$ whose valleys correspond to $V$. In addition, there exist $k + 2l + 1$ rectilinear polygons $P_1, \ldots, P_{k+2l+1}$ contained in $P$ such that*

1. $\cup_i P_i = P,\ i = 1, \ldots, k + 2l + 1.$
2. *$R$ coincides with the upper envelope of $R^*(P_i)$ for all $i = 1, \ldots, k + 2l + 1$.*

## 5 The Number of Realistic Roofs

We give an upper bound of the number of possible realistic roofs over $P$ in terms of $n$. For this, we need a few technical lemmas.

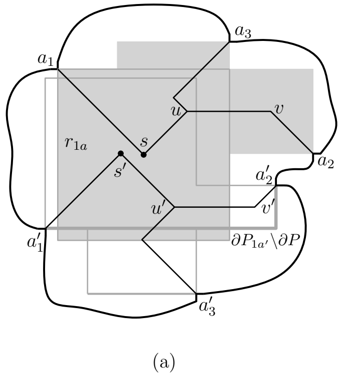


**Fig. 6.** (a) $r_{1a}$, $r_1$ of $uv$, meets $\partial P_{1a'} \backslash \partial P$. Between two peak points $s$ and $s'$, we cannot construct a roof face. (b) Between the peak point $s$ the open valley $uv$, we cannot construct a roof face.

**Lemma 9.** *Let $(a_1, a_2, a_3)$ be a candidate triple, where $a_1$ and $a_2$ have opposite orientations. Then $(a_1, a_2)$ is also a candidate pair.*

*Proof.* The candidate triple $(a_1, a_2, a_3)$ admits a half open valley $uv$. The free space of $uv$ contains $B_{a_1 a_2}$, so $a_1$ and $a_2$ admit an open valley $u'v'$ *related* to $uv$. ◻

**Lemma 10.** *Let $(a_1, a_2, a_3)$ be a candidate triple for a a half-open valley $uv$, where $a_1$ and $a_2$ have opposite orientations. If a candidate pair $(a_4, a_5)$ is compatible with $(a_1, a_2, a_3)$, then $(a_3, a_4, a_5)$ is not a candidate triple.*

*Proof.* Without loss of generality, assume that the three reflex vertices $a_1, a_2, a_3$ and the valley $uv$ located as in Figure 7(a). By Lemma 8, both $a_4$ and $a_5$ must be contained one of three rectilinear subpolygons of $P$, $P_1, P_2$ and $P_3$ defined by $uv$, as we did before. Assume to the contrary that $(a_3, a_4, a_5)$ is a candidate triple for a half-open valley $u'v'$.

*Case 1.* $a_4, a_5 \in \partial P_3$. There is only one possible configuration (Figure 7(b)). By some careful case analysis, we have $d_h(a_5, a_3) > d_h(a_4, a_3) > d_v(a_4, a_3)$, which makes $a_4$ be contained in the interior of the free space of $u'v'$.

*Case 2.* $a_4, a_5 \in \partial P_2$. There is no possible configuration.

*Case 3.* $a_4, a_5 \in \partial P_1$. There are two possible configurations. In case of Figure 7(c), we have $d_h(a_5, a_3) > d_h(a_1, a_3) > d_v(a_1, a_3)$, which makes $a_1$ be contained in the interior of the free space of $u'v'$. In case of Figure 7(d), we have $d_v(a_4, a_3) > d_h(a_1, a_3) > d_v(a_1, a_3)$, which again makes $a_1$ be contained in the interior of the free space of $u'v'$. ◻

Now we can find an upper bound of the number of realistic roofs over $P$.

**Theorem 2.** *Let $P$ be a rectilinear polygon with $n$ vertices. There are at most $1.3211^m \binom{m}{\lfloor \frac{m}{2} \rfloor}$ distinct realistic roofs over $P$, where $m = \frac{n-4}{2}$.*

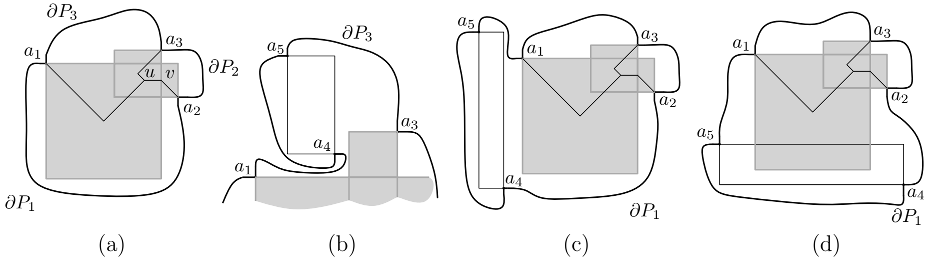


**Fig. 7.** Illustration of the proof of Lemma 10. Gray regions are free spaces.

*Proof.* Let $R$ be a realistic roof over $P$ with a half-open valley $uv$. By Lemma 9, we can get an open valley $u'v'$ related to $uv$. Therefore, we can get a new realistic roof by replacing $uv$ to $u'v'$. By repeating this process, we can get a realistic roof $R'$ which does not contain any half-open valleys. It means that for any realistic roof $R$ over $P$, there exist a unique realistic roof $R'$ which has no half-open valleys. We can get the number of distinct realistic roofs over $P$ with two steps: counting the number of realistic roofs $R'$ over $P$ which has no half-open valleys and counting the number of realistic roofs $R$ which can be transformed to each $R'$ by replacing its half-open valleys to related open valleys.

Ahn et al. [1,2] proved that the number of realistic roofs $R'$ over $P$ which has no half-open valleys is at most $\binom{m}{\lfloor \frac{m}{2} \rfloor}$, where $m = \frac{n-4}{2}$. We calculate the number of realistic roofs $R$ over $P$ corresponding to each $R'$. Suppose that $R'$ contains $k$ open valleys, $u_1v_1, u_2v_2, \ldots, u_kv_k$. $P$ has $m-2k$ reflex vertices that are not used to make open valleys. Let us call these reflex vertices as free vertices of $R'$. By Lemma 10, each of free vertices can make a half-open valley with at most one valley of $u_1v_1, u_2v_2, \ldots, u_kv_k$. Let $x_i$, $1 \leq i \leq k$, be the number of free vertices of $R'$ which can make a half-open valley with $u_iv_i$. Then the number of realistic roofs that can be transformed to $R'$ is at most $(x_1+1)(x_2+1)\ldots(x_k+1)$, where $x_1 + x_2 + \ldots + x_k \leq m - 2k$. From the inequality of arithmetic and geometric means, we can get $(x_1+1)(x_2+1)\ldots(x_k+1) \leq (\frac{x_1+x_2+\ldots+x_k+k}{k})^k \leq (\frac{m-k}{k})^k = ((\frac{m}{k}-1)^{\frac{k}{m}})^m$. For a positive real number $x$, $\sup\{(x-1)^{\frac{1}{x}}\} \approx 1.3210997\ldots$, so $((\frac{m}{k}-1)^{\frac{k}{m}})^m < 1.3211^m$. Therefore, we can get at most $1.3211^m$ different realistic roofs over $P$ corresponding to each $R'$, and the total number of distinct realistic roofs over $P$ is at most $1.3211^m\binom{m}{\lfloor \frac{m}{2} \rfloor}$. ◻

## 6 Algorithm

In this section, we will present an algorithm that generates all possible realistic roofs over given rectilinear polygon $P$. Ahn et al. [1,2] suggested an efficient algorithm that generates all realistic roofs which do not have half-open valleys. Let us call the algorithm *Ahn's algorithm*. *Ahn's algorithm* uses $O(n^4)$ time as preprocessing and generates realistic roofs one by one in $O(1)$ time each.

We also use $O(n^4)$ preprocessing time. $P$ has $O(n^3)$ triples and $O(n^2)$ pairs of reflex vertices, and checking whether each triple and pair is a candidate triple or candidate pair takes $O(n)$ time. After $O(n^4)$ time, we can get all candidate triples and pairs of $P$. Create an empty list of reflex vertices for each candidate pair and add a reflex vertex $a_i$ to a candidate pair's list if $a_i$ and the candidate pair form a candidate triple.

Our algorithm works as follows. Run *Ahn's algorithm* and get a realistic roof $R$ with $k$ open valleys $u_1v_1, \ldots, u_kv_k$. A candidate pair $(a_i, a'_i)$ corresponding to $u_iv_i$, $1 \leq i \leq k$, has a list of reflex vertices and let $x_i$ be a reflex vertex chosen from the list. If we do not choose any vertex from the list of $(a_i, a'_i)$, let $x_i = \emptyset$. For the chosen vertices $x_1, \ldots, x_k$, check whether the set of candidate pairs and triples $V = \{(a_1, a'_1, x_1), \ldots, (a_k, a'_k, x_k)\}$ is a compatible set of $P$ where $(a_i, a'_i, \emptyset) = (a_i, a'_i)$. If $(x_1, \ldots, x_k) = (\emptyset, \ldots, \emptyset)$, $R$ is the realistic roof whose valleys correspond to $V$. By changing $(x_1, \ldots, x_k)$ one by one, checking the compatibility of $V$ takes $O(k)$ time. Suppose that $(\ldots, x_i, \ldots)$ is changed to $(\ldots, x'_i, \ldots)$. We already know compatibilities between valleys which are induced by $(a_j, a'_j, x_j)$ for $j = 1, \ldots, k$ and keep the total number of "conflicts" between the valleys. Check compatibilities between the valley induced by $(a_i, a'_i, x_i)$ and the others, and decrease the total number of conflicts when it is not compatible with others. Next, check compatibilities between the valley induced by $(a_i, a'_i, x'_i)$ and the others, and increase the total number of conflicts when it is not compatible with others. After that, if the total number of conflicts is zero, then the set $V$ is compatible for $P$. Therefore, our algorithm finds all realistic roofs correspond to each $R$ in $O(m1.3211^m)$ time.

**Theorem 3.** *Given a rectilinear polygon $P$ with $n$ vertices, $m$ of which are reflex vertices, after $O(n^4)$-time preprocessing, all the compatible sets of $P$ can be enumerated in $O(m1.3211^m\binom{m}{\lfloor \frac{m}{2} \rfloor})$.*

## References


1. Ahn, H.-K., Bae, S.W., Knauer, C., Lee, M., Shin, C.-S., Vigneron, A.: Generating realistic roofs over a rectilinear polygon. In: Asano, T., Nakano, S.-I., Okamoto, Y., Watanabe, O. (eds.) ISAAC 2011. LNCS, vol. 7074, pp. 60–69. Springer, Heidelberg (2011)
2. Ahn, H.-K., Bae, S.W., Knauer, C., Lee, M., Shin, C.-S., Vigneron, A.: Realistic roofs over a rectilinear polygon. Submitted to a Journal. Personal Communication
3. Aichholzer, O., Albertsa, D., Aurenhammer, F., Gärtner, B.: A novel type of skeleton for polygons. J. Universal Comput. Sci. 1, 752–761 (1995)
4. Aichholzer, O., Aurenhammer, F.: Straight skeletons for general polygonal figures in the plane. In: Cai, J.-Y., Wong, C.K. (eds.) COCOON 1996. LNCS, vol. 1090, pp. 117–226. Springer, Heidelberg (1996)
5. Brenner, C.: Interactive modelling tools for 3D building reconstruction. In: Fritsch, D., Spiller, R. (eds.) Photogrammetric Week 1999, pp. 23–34 (1999)
6. Brenner, C.: Towards fully automatic generation of city models. Int. Archives of Photogrammetry and Remote Sensing XXXIII(pt. B3), 85–92 (2000)

7. Khoshelham, K., Li, Z.L.: A split-and-merge technique for automated reconstruction of roof planes. Photogrammetric Engineering and Remote Sensing 71(7), 855–863 (2005)
8. Krauß, T., Lehner, M., Reinartz, P.: Generation of coarse 3D models of urban areas from high resolution stereo satellite images. Int. Archives of Photogrammetry and Remote Sensing XXXVII, 1091–1098 (2008)
9. Laycock, R.G., Day, A.M.: Automatically generating large urban environments based on the footprint data of buildings. In: Proc. 8th ACM Sympos. Solid Model. Appl., pp. 346–351 (2003)
10. Sohn, G., Huang, X.F., Tao, V.: Using a binary space partitioning tree for reconstructing polyhedral building models from airborne lidar data. Photogrammetric Engineering and Remote Sensing 74(11), 1425–1440 (2008)