Approximating a Motorcycle Graph by a Straight Skeleton

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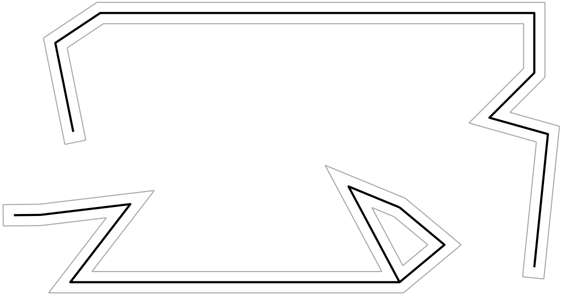
Introduction to straight skeletons

- [Aichholzer and Aurenhammer, 1996]: straight skeleton of a PSLG G
- Defined by the **propagation of a wavefront**.



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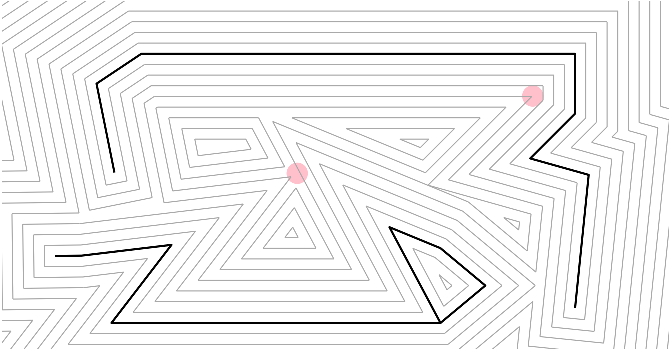

Introduction to straight skeletons

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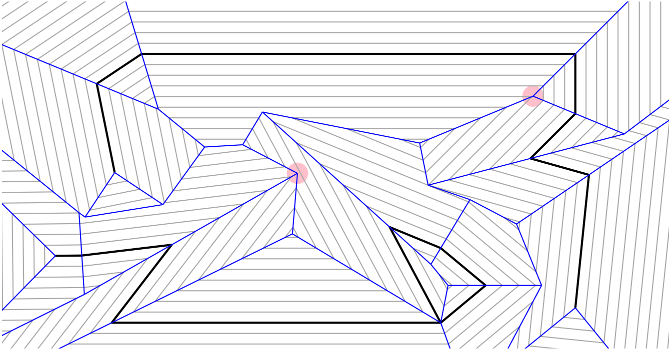
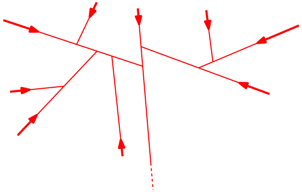
- Defined by the **propagation of a wavefront**.
  - edge event
  - split event
- Straight skeleton $\mathcal{S}(G)$: set of loci that are traced out by wavefront vertices.
  - Edges of $\mathcal{S}(G)$ are called *arcs*.
  - *Reflex arcs* are traced by reflex wavefront vertices.



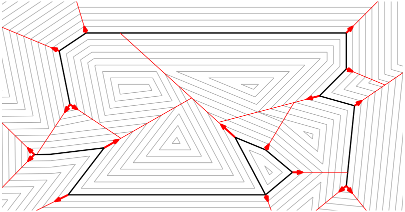
- Introduced by [Eppstein and Erickson, 1999].
- A **motorcycle** is a point moving with constant velocity, starting from a start point.
- **Motorcycle graph**:
  - Given n motorcycles $m_1, \ldots, m_n$.
  - Motorcycles leave behind a trace while driving.
  - If motorcycle reaches other motorcycle's trace, it *crashes*.
  - Motorcycle graph $\mathcal{M}(m_1, \ldots, m_n)$: the arrangement of all motorcycle traces.



Geometric relationship: motorcycle graph and straight skeleton

Consider a PSLG G.

- Define for each reflex wavefront vertex a motorcycle with the same velocity.
- The edges of G are walls.
- Denote the resulting motorcycle graph by $\mathcal{M}(G)$.
- *Assumption:* no two motorcycle crash simultaneously into each other.



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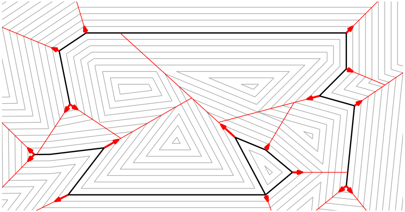

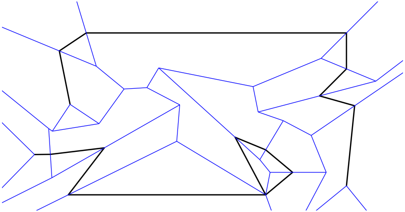

Theorem ([Cheng and Vigneron, 2007], [Huber and Held, 2011])

The motorcycle graph $\mathcal{M}(G)$ covers the reflex arcs of the straight skeleton $\mathcal{S}(G)$.

Applications

Fruitful geometric relationship between $\mathcal{M}(G)$ and $\mathcal{S}(G)$:

- Alternative characterization of straight skeletons.
- Theoretical $O(n\sqrt{n}\log^2 n)$ straight-skeleton algorithm for non-degenerate polygons with holes [Cheng and Vigneron, 2007].
- Bone: a fast and implementable straight-skeleton algorithm for PSLGs that exhibits an $O(n \log n)$ runtime in practice [Huber and Held, 2011].

Idea

Further investigate the geometric relationship between $\mathcal{M}(G)$ and $\mathcal{S}(G)$!

Roadmap

So far: reducing the computation of $\mathcal{S}(G)$ to $\mathcal{M}(G)$.

New questions:

1. Can we reduce the computation of $\mathcal{M}(G)$ to $\mathcal{S}(G)$?
   - ... to transfer lower runtime bounds for $\mathcal{S}(G)$.
   - ... to learn more about the relative complexity.

Roadmap

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New questions:

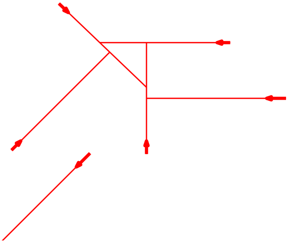
1. Can we reduce the computation of $\mathcal{M}(G)$ to $\mathcal{S}(G)$?
   - ... to transfer lower runtime bounds for $\mathcal{S}(G)$.
   - ... to learn more about the relative complexity.
2. We know: $\mathcal{M}(G)$ covers reflex arcs of $\mathcal{S}(G)$.
   - Given motorcycles $m_1, \dots, m_n$, can we construct a G such that parts of $\mathcal{S}(G)$ approximate $\mathcal{M}(m_1, \dots, m_n)$ up to any given tolerance?
     - ... to learn more on the geometric relationship between $\mathcal{M}(G)$ and $\mathcal{S}(G)$.

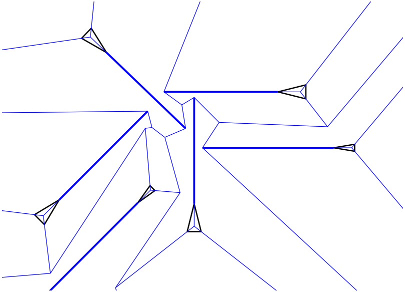
Approximating $\mathcal{M}(m_1, \ldots, m_n)$

Question

Can we find a PSLG G such that parts of $S(G)$ and $\mathcal{M}(m_1, \ldots, m_n)$ cover each other up to any given tolerance?

- Denote by p_i the start point and by v_i the velocity of m_i.
  - $\rightarrow$ Place isosceles triangle Δ_i at p_i and corresponding interior angle α_i.
  - How can we make the gaps between the traces and the reflex arcs small?





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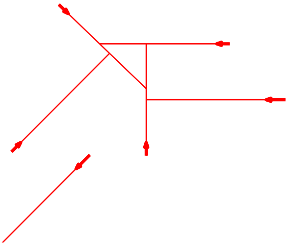
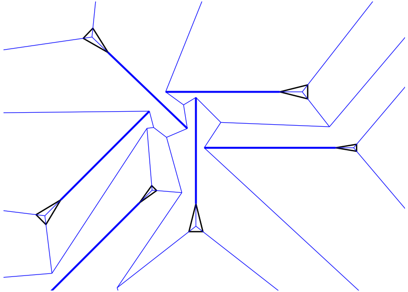
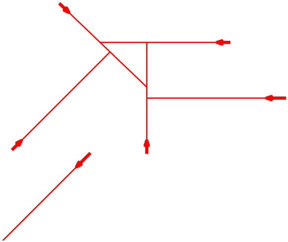
  - $\rightarrow$ Place isosceles triangle Δ_i at p_i and corresponding interior angle α_i.
  - How can we make the gaps between the traces and the reflex arcs small?
- General observation: faster motorcycles lead to smaller gaps.
  - Multiplying all v_i with a constant $\lambda > 0$ leaves $\mathcal{M}(m_1, \ldots, m_n)$ invariant!

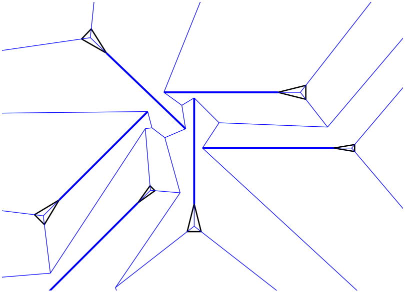




Question

Can we always find a λ large enough such that the gaps become arbitrarily small?





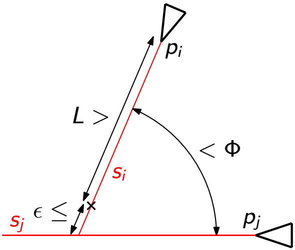
Approximating $\mathcal{M}(m_1, \ldots, m_n)$

- We denote by s_i the trace of m_i.

Lemma

Let m_i crash into s_j. Then s_i covers the reflex arc incident to p_i up to a gap size of at most ϵ if

$$\lambda \geq \frac{1}{\min_k |v_k| \sin \Phi} \cdot \max \left\{ 2, \frac{L}{\min\{\mu/4, \epsilon\}} \right\}.$$



Where

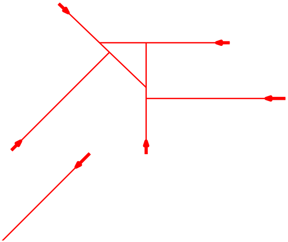
- $\overrightarrow{s_i}$ ...infinite track of m_i.
- $\mu := \frac{1}{3} \min_{1 \leq i,j,k \leq n} \mathbb{R}^+ \cap \{ d(\overrightarrow{s_i}, p_j), d(\overrightarrow{s_i}, \overrightarrow{s_j} \cap \overrightarrow{s_k}) \}$.
- $\varphi_{i,j} = \varphi_{j,i}$ ...non-oriented angle between v_i and v_j.
- $\Phi := \min_{1 \leq i < j \leq n} \mathbb{R}^+ \cap \{ \varphi_{i,j}, \pi - \varphi_{i,j} \}$.
- $L := \max_{1 \leq i,j \leq n} d(p_i, \overrightarrow{s_i} \cap \overrightarrow{s_j})$.

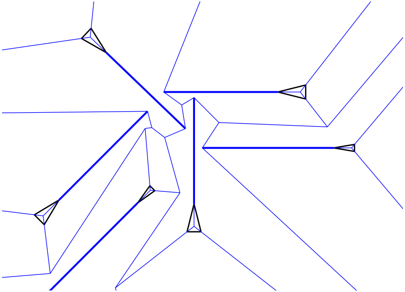
Computing $\mathcal{M}(m_1, \ldots, m_n)$

Question

Can we compute $\mathcal{M}(m_1, \ldots, m_n)$ using a straight skeleton?

- Still, some gaps remain.
- How to algorithmically decide which motorcycle crashes into which trace?





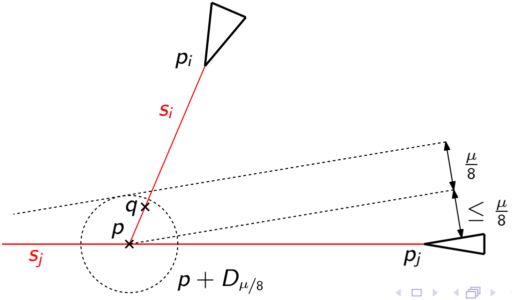
- u_i ... the reflex wavefront vertex emanating from p_i.

Lemma

Consider $\mathcal{S}(G)$ with

$$\lambda \geq \frac{\max\left\{2, \frac{8L}{\mu}\right\}}{\min_k |v_k| \cdot \sin \Phi}.$$

Then m_i crashes into s_j if and only if u_i leads to a split event with a wavefront emanated by Δ_j until time $\mu/4$.



Computing $\mathcal{M}(m_1, \ldots, m_n)$

Algorithm

1. Compute μ, L, Φ.
2. Generate G and compute $\mathcal{S}(G)$.
3. Apply the previous lemma in order to decide for each motorcycle whether it crashed, and into which trace.

An application: P-completeness of straight skeleton

- P-completeness w.r.t. NC-reductions:
  - NC: solvable in poly-logarithmic time using a polynomial number of processors.
  - A P-complete problem cannot be efficiently parallelized, unless $\mathrm{NC} = \mathrm{P}$.
  - $\mathrm{LOGSPACE} \subseteq \mathrm{NC}$

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Lemma

The construction of the straight skeleton of PSLGs G (resp. polygons with holes) is P-complete under $\mathrm{LOGSPACE}$-reductions.

- Has already been mentioned by [Eppstein and Erickson, 1999] without a proof. (Referring to similar arguments as for the motorcycle graph.)

Our proof:

- Determine L, Φ, μ for the motorcycle graphs generated by Eppstein and Erickson.
- Apply the algorithm mentioned before.

Summary

- Motorcycle graphs can be approximated arbitrarily well using straight skeletons.
- We have an algorithm to determine the motorcycle graph by employing the straight skeleton.
- Using this we obtain a proof for the P-completeness of straight skeletons of polygons with holes.
  - Consequently, straight skeleton algorithms cannot be parallelized efficiently to run in NC time, unless $\mathrm{NC} = \mathrm{P}$.

Finish

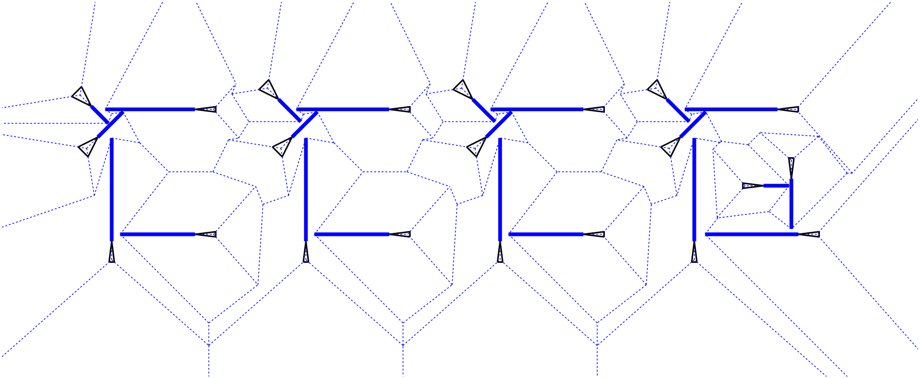

Figure: "CCCG" approximated by a motorcycle graph, approximated by a straight skeleton. The latter is computed using our straight-skeleton code BONE.

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P-completeness for polygons?

- The proof does not extend to simple polygons.
- Need a polygon that connects the start points of $m, m_1, \ldots, m_4$.
- Where to go through the square without knowing which motorcycle crashes in which trace?