# What makes a Tree a Straight Skeleton?


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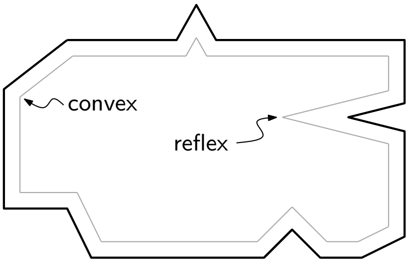
# Straight skeletons: an introduction

- Introduced for simple polygons $P$ in [Aichholzer et al., 1995].
- Definition based on **wavefront propagation process**:



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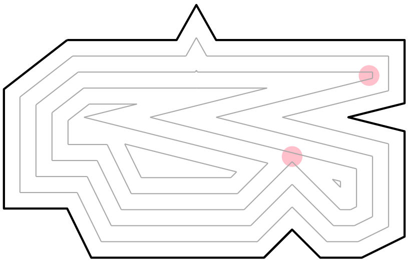

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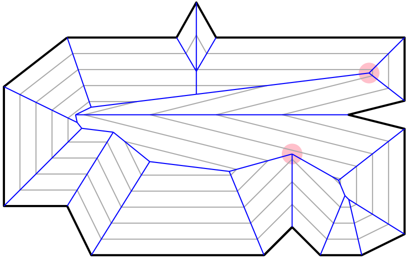
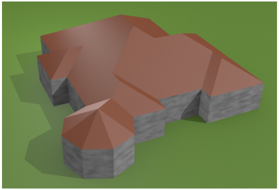
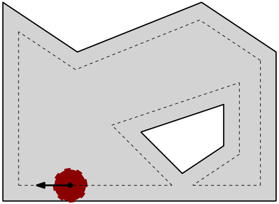
- Definition based on **wavefront propagation process**:
  - edge events,
  - split events.
- Straight skeleton $\mathcal{S}(P)$: set of loci traced out by wavefront vertices.
  - $\mathcal{S}(P)$ partitions $P$ into straight-skeleton faces.



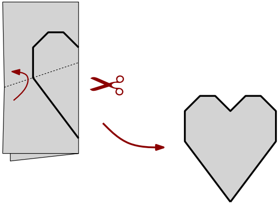
# Applications



Roof construction



Tool path generation



Fold-and-cut problem

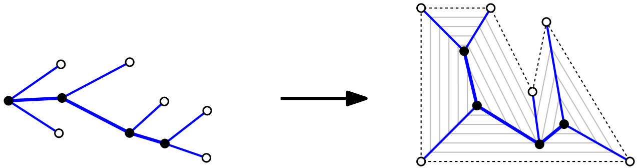
. . . and many more.

# An inverse straight-skeleton problem

We are given:

- a tree.

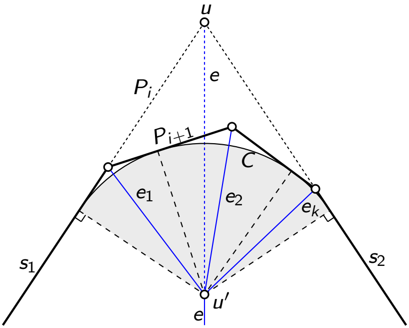
Can we find a polygon $P$ whose straight-skeleton $\mathcal{S}(P)$ has the same graph structure?



# An inverse straight-skeleton problem

## Theorem

*For any tree $T$, whose inner vertices have at least degree 3, there exists a feasible (convex) polygon $P$ such that $\mathcal{S}(P)$ possesses the same graph structure as $T$.*

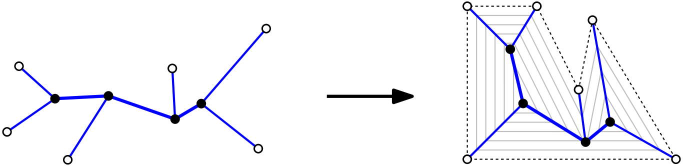


# An semi-geometric inverse straight-skeleton problem

We are given:

- a tree (topologically),
- the lengths of the edges,
- at each vertex the circular order of the incident edges.

Can we find a polygon $P$ whose straight-skeleton $\mathcal{S}(P)$ matches these requirements?



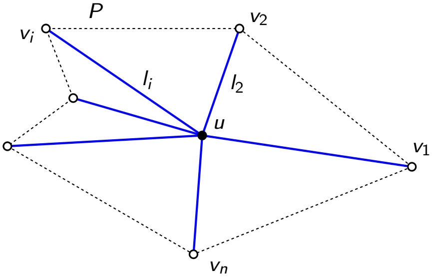
# Notations

- We call the set of geometric graphs with given topology, edge lengths and incidence orders an abstract geometric graph $\mathcal{G}$.
- A polygon $P$ is suitable for $\mathcal{G}$ if $\mathcal{S}(P) \in \mathcal{G}$.
- A $\mathcal{G}$ is feasible if it has a suitable polygon.

- Which $\mathcal{G}$ are feasible?
- If $\mathcal{G}$ is feasible, are the suitable polygons unique?
- How to construct suitable polygons?

# Star graphs $S_n$: introduction

- Let us start with simple trees: star graphs $S_n$.
  - A vertex $u$ adjacent to $n$ terminal vertices $v_1, \ldots, v_n$.
  - We denote by $l_i$ the length of $uv_i$. W.l.o.g. let $l_1 = \max_i l_i$.



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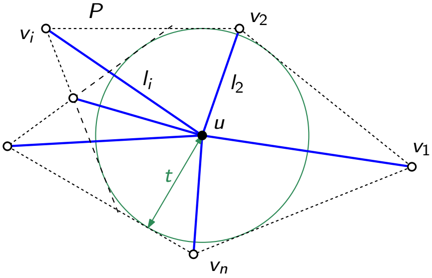
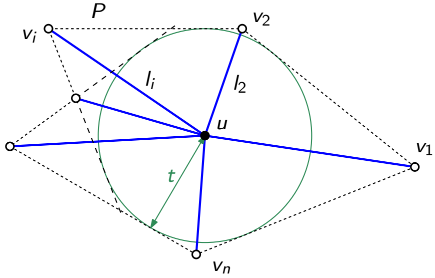
  - We denote by $l_i$ the length of $uv_i$. W.l.o.g. let $l_1 = \max_i l_i$.
- If $P$ is suitable then $u$ has equal orthogonal distance to all polygon edges.
  - Hence, there is a tangential circle with some radius $t$.



# Star graphs $S_n$: introduction




## Observation

*If $P$ is suitable for $S_n$ then*

1. *two consecutive vertices cannot be both reflex,*
2. $l_i < l_{i\pm1}$ *for a reflex $v_i$,*
3. *the edges of $P$ have equal orthogonal distance $t$ to $u$, with $t \leq \min_i l_i$.*

# Constructing suitable polygons for $S_n$

Is there a suitable polygon for $S_n$ for a given convexity/reflexivity assignment $A$ to its vertices?

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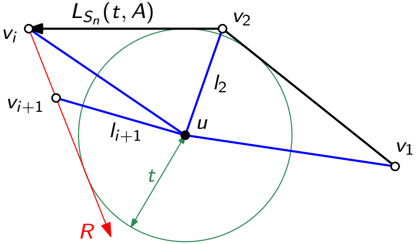
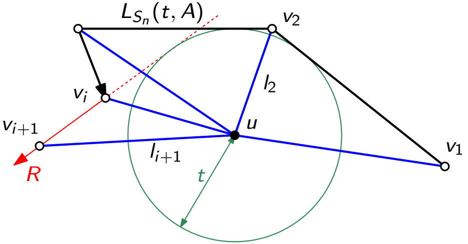
We construct the following polyline $L_{S_n}(t, A)$:

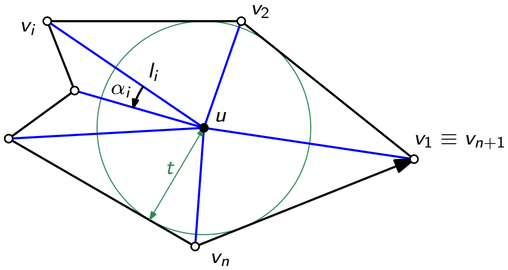
- Place a circle $C$ with radius $t$ and center $u = (0, 0)$ and a vertex $v_1$ at $(l_1, 0)$.
- We incrementally construct $v_2, \ldots, v_{n+1}$:
  - Shoot a tangential ray $R$ from $v_i$ right to $C$.
  - Place $v_{i+1}$ on the ray at desired distance $l_{1+i \bmod n}$ to $u$.



# Constructing suitable polygons for $S_n$

- **Basic idea:** If $L_{S_n}(t, A)$ is closed and simple then $L_{S_n}(t, A)$ forms a suitable polygon.
- $L_{S_n}(t, A)$ is closed and simple if and only if $\alpha_A(t) := \sum_{i=1}^{n} \alpha_i = 2\pi$.



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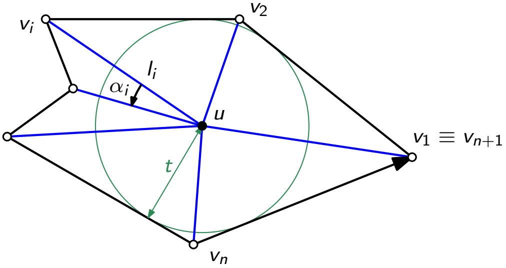


## Lemma

$$\alpha_A(t) = 2 \sum_{\substack{i=1 \\ v_i \text{ convex}}}^{n} \arccos \frac{t}{l_i} - 2 \sum_{\substack{i=1 \\ v_i \text{ reflex}}}^{n} \arccos \frac{t}{l_i} \,. \tag{1}$$

# Is a star graph $S_n$ feasible?

**Lemma**

*A suitable convex polygon for a star graph $S_n$ exists if and only if $\sum_i \arccos \frac{\min_i l_i}{l_i} \leq \pi$. If a suitable convex polygon exists then it is unique.*

Proof idea: Show that a $t \in (0, \min_i l_i]$ exists with $\alpha_A(t) = 2\pi$.

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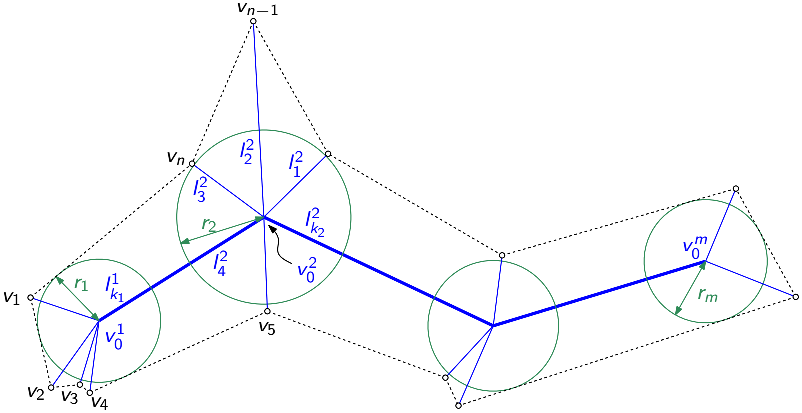
## Lemma

- *There exist infeasible star graphs $S_5$.*
- *There exist star graphs $S_5$ for which multiple suitable polygons exist.*

# Caterpillar graphs: notations

- A caterpillar graph $G$ becomes a path (backbone) if all leaves are removed.
  - Backbone vertices are denoted by $v_0^1, \ldots, v_0^m$.




Can we express the sum of inner angles of $P$ as a function of one parameter?

# Caterpillar graphs: geometric properties

**Lemma**

*The radii $r_2, \ldots, r_m$ for some given caterpillar graph $G$ are determined by $r_1$ according to the following recursions, for $1 \leq i < m$:*

$$r_{i+1} = r_i + l^i_{k_i} \sin \beta_i \tag{2}$$

$$\beta_i = \beta_{i-1} + (1 - {}^{k_i}\!/_2)\pi + \sum_{\substack{j=1 \\ v^i_j \neq v^{i-1}_0}}^{k_i - 1} \begin{cases} \arcsin \frac{r_i}{l^i_j} & \text{if } v^i_j \text{ is convex} \\ \pi - \arcsin \frac{r_i}{l^i_j} & \text{if } v^i_j \text{ is reflex} \end{cases} \tag{3}$$

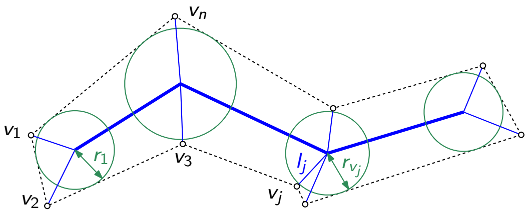
*For $i = 1$ we define that $\beta_0 = 0$ and $v^1_j \neq v^0_0$ being true for all $1 \leq j < k_1$.*

# Caterpillar graphs: feasibility and suitable polygons

## Corollary

*The sum of the inner angles of $P$ with convexity assignment $A$ is a function*

$$\alpha_A(r_1) = 2\sum_{j=1}^{n} \begin{cases} \arcsin \frac{r_{v_j}}{l_j} & \text{if } v_j \text{ is convex} \\ \pi - \arcsin \frac{r_{v_j}}{l_j} & \text{if } v_j \text{ is reflex} \end{cases} . \tag{4}$$



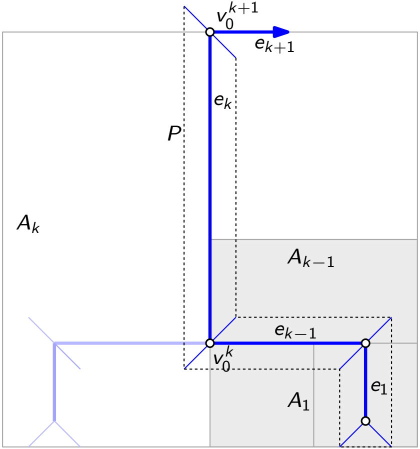

## Lemma

*There is only a finite number of suitable polygons for a caterpillar graph.*

# How many suitable polygons can exist?

## Lemma

*There exists a caterpillar graph with $3m$ vertices having $2^{m-2}$ suitable polygons.*



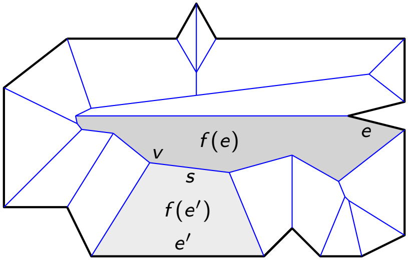
# Finish

# Bibliography

Aichholzer, O., Alberts, D., Aurenhammer, F., and Gärtner, B. (1995).
A novel type of skeleton for polygons.
*J. Universal Comp. Sci.*, 1(12):752–761.

# Straight skeletons: basic geometric properties

- $P$ is tessellated into faces.
  - Each face $f(e)$ belongs to an edge $e$.
- Every straight-skeleton edge $s$ is on the boundary of two faces, $f(e)$ and $f(e')$, and lies on the bisector of $e$ and $e'$.
- A straight-skeleton vertex $v$ on the boundary of faces $f(e_1), \ldots, f(e_k)$ has equal orthogonal distance to $e_1, \ldots, e_k$.



# Infeasible star graph $S_5$

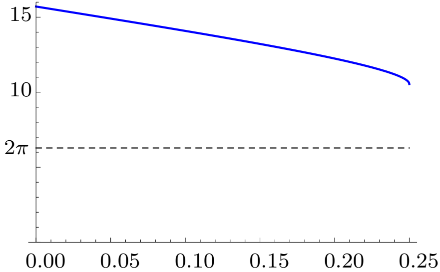



Figure : The sum $\sum_i \alpha_i$ for all $t \in (0, \min_i l_i]$, where $l_1 = \cdots = l_4 = 1$ and $l_5 = 0.25$. Left: $v_5$ is convex. Right: $v_5$ is reflex.

# Multiple feasible polygons for $S_5$

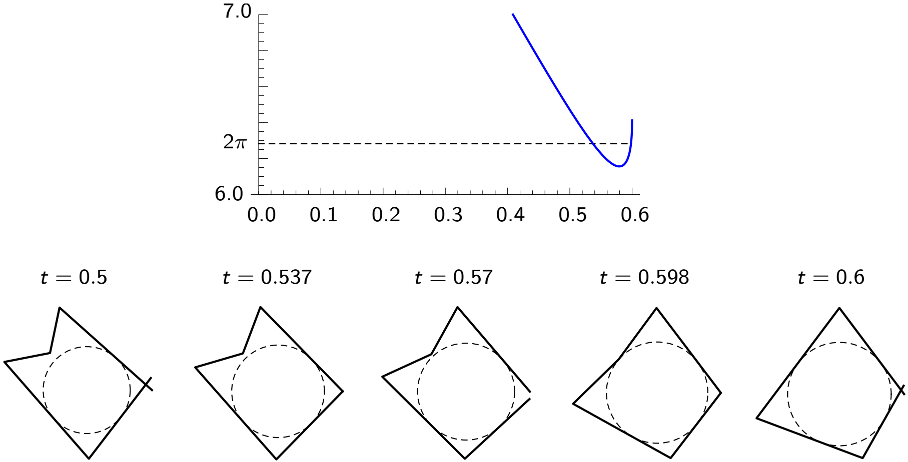



Figure : Edge lengths $l_1 = 0.75$, $l_2 = 1$, $l_3 = 0.6$, $l_4 = 1$, $l_5 = 0.79$. All vertices are convex, except for $v_3$. Top: $\sum_i \alpha_i$ evaluates to $2\pi$ for two different values of $t$. Bottom: The result of our construction scheme for a sequence of different values of $t$.