What makes a Tree a Straight Skeleton?

O. Aichholzer\textsuperscript{1} H. Cheng\textsuperscript{2} S. L. Devadoss\textsuperscript{3} T. Hackl\textsuperscript{1} S. Huber\textsuperscript{4} B. Li\textsuperscript{3} A. Risteski\textsuperscript{5}

\textsuperscript{1}Graz University of Technology Austria
\textsuperscript{2}University of Arizona Tuscon, AZ, USA
\textsuperscript{3}Williams College Williamstown, MA, USA
\textsuperscript{4}Universität Salzburg Austria
\textsuperscript{5}Princeton University Princeton, NJ, USA

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Straight skeletons: an introduction

- Introduced for simple polygons $P$ in [Aichholzer et al., 1995].
- Definition based on *wavefront propagation process*:

![Diagram of a straight skeleton with convex and reflex vertices]
Straight skeletons: an introduction

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  - edge events,
  - split events.
Straight skeletons: an introduction

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- Definition based on **wavefront propagation process**:
  - edge events,
  - split events.
- **Straight skeleton** $S(P)$: set of loci traced out by wavefront vertices.
  - $S(P)$ partitions $P$ into straight-skeleton faces.
Applications

Roof construction

Tool path generation

Fold-and-cut problem

...and many more.
An inverse straight-skeleton problem

We are given:
- a tree.

Can we find a polygon $P$ whose straight-skeleton $S(P)$ has the same graph structure?
An inverse straight-skeleton problem

**Theorem**

For any tree $T$, whose inner vertices have at least degree 3, there exists a feasible (convex) polygon $P$ such that $S(P)$ possesses the same graph structure as $T$. 
An semi-geometric inverse straight-skeleton problem

We are given:
- a tree (topologically),
- the lengths of the edges,
- at each vertex the circular order of the incident edges.

Can we find a polygon $P$ whose straight-skeleton $S(P)$ matches these requirements?
Notations

- We call the set of geometric graphs with given topology, edge lengths and incidence orders an abstract geometric graph \( G \).
- A polygon \( P \) is suitable for \( G \) if \( S(P) \in G \).
- A \( G \) is feasible if it has a suitable polygon.

- Which \( G \) are feasible?
- If \( G \) is feasible, are the suitable polygons unique?
- How to construct suitable polygons?
Let us start with simple trees: star graphs $S_n$.

- A vertex $u$ adjacent to $n$ terminal vertices $v_1, \ldots, v_n$.
- We denote by $l_i$ the length of $uv_i$. W.l.o.g. let $l_1 = \max_i l_i$. 

If $P$ is suitable then $u$ has equal orthogonal distance to all polygon edges. Hence, there is a tangential circle with some radius $t$. 

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[Diagram of a star graph $S_n$ with a vertex $u$ adjacent to $n$ terminal vertices $v_1, \ldots, v_n$, showing lengths $l_i$ and a tangential circle $P$.]
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Observation

If $P$ is suitable for $S_n$ then

1. two consecutive vertices cannot be both reflex,
2. $l_i < l_{i±1}$ for a reflex $v_i$,
3. the edges of $P$ have equal orthogonal distance $t$ to $u$, with $t \leq \min_i l_i$. 
Constructing suitable polygons for $S_n$

Is there a suitable polygon for $S_n$ for a given convexity/reflexivity assignment $A$ to its vertices?
Constructing suitable polygons for $S_n$

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We construct the following polyline $L_{S_n}(t, A)$:

- Place a circle $C$ with radius $t$ and center $u = (0, 0)$ and a vertex $v_1$ at $(l_1, 0)$.
- We incrementally construct $v_2, \ldots, v_{n+1}$:
  - Shoot a tangential ray $R$ from $v_i$ right to $C$.
  - Place $v_{i+1}$ on the ray at desired distance $l_{1+i \mod n}$ to $u$. 

Aichholzer, Cheng, Devadoss, Hackl, Huber, Li, Risteski: What makes a Tree a Straight Skeleton?
Constructing suitable polygons for $S_n$

- **Basic idea:** If $L_{S_n}(t, A)$ is closed and simple then $L_{S_n}(t, A)$ forms a suitable polygon.
- $L_{S_n}(t, A)$ is closed and simple if and only if $\alpha_A(t) := \sum_{i=1}^{n} \alpha_i = 2\pi$. 

$$v_1 \equiv v_{n+1}$$
Constructing suitable polygons for $S_n$

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- $L_{S_n}(t, A)$ is closed and simple if and only if $\alpha_A(t) := \sum_{i=1}^{n} \alpha_i = 2\pi$.

\[ \alpha_A(t) = 2 \sum_{i=1}^{n} \begin{cases} \arccos \frac{t}{l_i} & \text{for convex } v_i \\ -2 \sum_{i=1}^{n} \arccos \frac{t}{l_i} & \text{for reflex } v_i \end{cases} \]  

(1)
Is a star graph $S_n$ feasible?

**Lemma**

A suitable convex polygon for a star graph $S_n$ exists if and only if \[ \sum_i \arccos \frac{\min_i l_i}{l_i} \leq \pi. \] If a suitable convex polygon exists then it is unique.

Proof idea: Show that a $t \in (0, \min_i l_i]$ exists with $\alpha_A(t) = 2\pi$. 

Aichholzer, Cheng, Devadoss, Hackl, Huber, Li, Risteski: *What makes a Tree a Straight Skeleton?*
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**Lemma**

- There exist infeasible star graphs $S_5$.
- There exist star graphs $S_5$ for which multiple suitable polygons exist.
Caterpillar graphs: notations

- A caterpillar graph $G$ becomes a path (backbone) if all leaves are removed.
  - Backbone vertices are denoted by $v_0^1, \ldots, v_0^m$.

Can we express the sum of inner angles of $P$ as a function of one parameter?
Lemma

The radii \( r_2, \ldots, r_m \) for some given caterpillar graph \( G \) are determined by \( r_1 \) according to the following recursions, for \( 1 \leq i < m \):

\[
\begin{align*}
    r_{i+1} &= r_i + l^i_{k_i} \sin \beta_i \\
    \beta_i &= \beta_{i-1} + (1 - k_i/2)\pi + \left\{\begin{array}{ll}
        \arcsin \frac{r_i}{l^i_j} & \text{if } v^i_j \text{ is convex} \\
        \pi - \arcsin \frac{r_i}{l^i_j} & \text{if } v^i_j \text{ is reflex}
    \end{array}\right.
\end{align*}
\]

For \( i = 1 \) we define that \( \beta_0 = 0 \) and \( v^1_j \neq v^0_0 \) being true for all \( 1 \leq j < k_1 \).
Corollary

The sum of the inner angles of $P$ with convexity assignment $A$ is a function

$$\alpha_A(r_1) = 2 \sum_{j=1}^{n} \begin{cases} \arcsin \frac{r_{vj}}{l_j} & \text{if } v_j \text{ is convex} \\ \pi - \arcsin \frac{r_{vj}}{l_j} & \text{if } v_j \text{ is reflex} \end{cases}.$$  \hspace{1cm} (4)

Lemma

There is only a finite number of suitable polygons for a caterpillar graph.
How many suitable polygons can exist?

**Lemma**

There exists a caterpillar graph with $3m$ vertices having $2^{m-2}$ suitable polygons.
Finish
Straight skeletons: basic geometric properties

- $P$ is tessellated into faces.
  - Each face $f(e)$ belongs to an edge $e$.
- Every straight-skeleton edge $s$ is on the boundary of two faces, $f(e)$ and $f(e')$, and lies on the bisector of $e$ and $e'$.
- A straight-skeleton vertex $v$ on the boundary of faces $f(e_1), \ldots, f(e_k)$ has equal orthogonal distance to $e_1, \ldots, e_k$. 

![Diagram of a straight skeleton with faces and edges labeled](image)
Infeasible star graph $S_5$

Figure: The sum $\sum_i \alpha_i$ for all $t \in (0, \min_i l_i]$, where $l_1 = \cdots = l_4 = 1$ and $l_5 = 0.25$. Left: $v_5$ is convex. Right: $v_5$ is reflex.
Multiple feasible polygons for $S_5$

Figure: Edge lengths $l_1 = 0.75$, $l_2 = 1$, $l_3 = 0.6$, $l_4 = 1$, $l_5 = 0.79$. All vertices are convex, except for $v_3$. Top: $\sum_i \alpha_i$ evaluates to $2\pi$ for two different values of $t$. Bottom: The result of our construction scheme for a sequence of different values of $t$. 