

What makes a Tree a Straight Skeleton?

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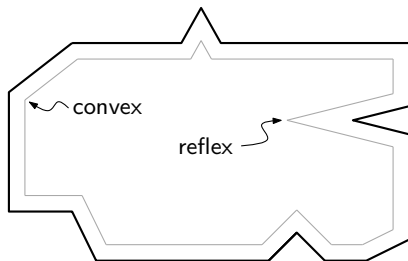
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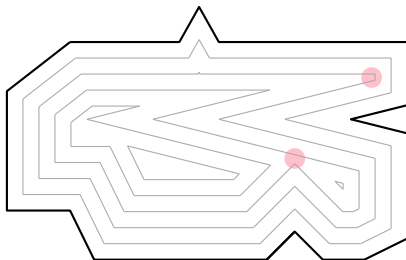
Straight skeletons: an introduction

- ▶ Introduced for simple polygons P in [Aichholzer et al., 1995].
- ▶ Definition based on **wavefront propagation process**:



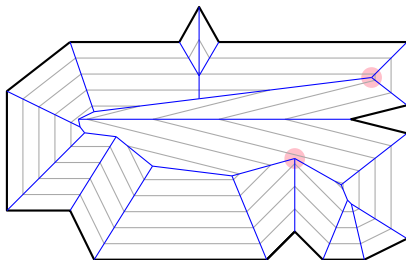
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- ▶ Definition based on **wavefront propagation process**:
 - ▶ edge events,
 - ▶ split events.

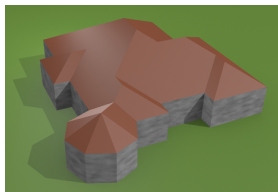


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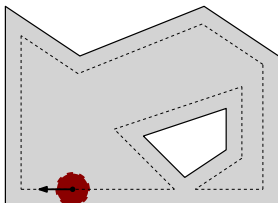
- ▶ Introduced for simple polygons P in [Aichholzer et al., 1995].
- ▶ Definition based on **wavefront propagation process**:
 - ▶ edge events,
 - ▶ split events.
- ▶ Straight skeleton $\mathcal{S}(P)$: set of loci traced out by wavefront vertices.
 - ▶ $\mathcal{S}(P)$ partitions P into straight-skeleton faces.



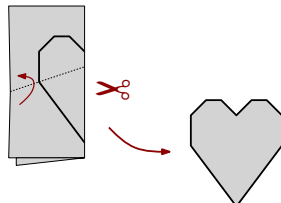
Applications



Roof construction



Tool path generation



Fold-and-cut problem

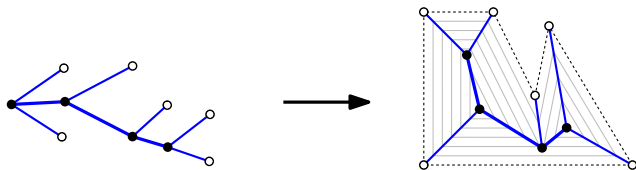
...and many more.

An inverse straight-skeleton problem

We are given:

- ▶ a tree.

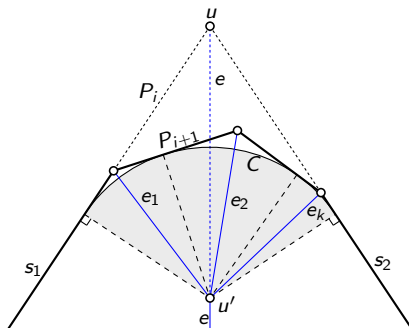
Can we find a polygon P whose straight-skeleton $\mathcal{S}(P)$ has the same graph structure?



An inverse straight-skeleton problem

Theorem

For any tree T , whose inner vertices have at least degree 3, there exists a feasible (convex) polygon P such that $S(P)$ possesses the same graph structure as T .

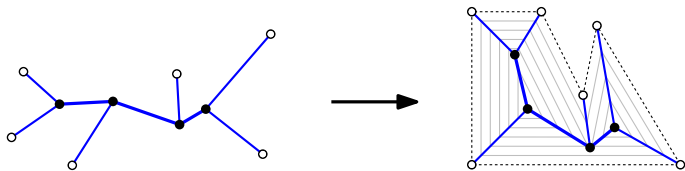


An semi-geometric inverse straight-skeleton problem

We are given:

- ▶ a tree (topologically),
- ▶ the lengths of the edges,
- ▶ at each vertex the circular order of the incident edges.

Can we find a polygon P whose straight-skeleton $\mathcal{S}(P)$ matches these requirements?

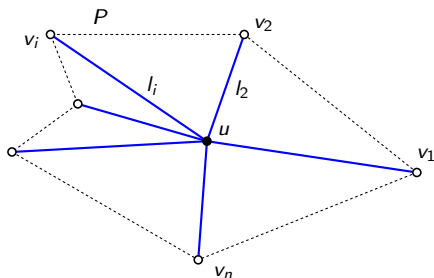


Notations

- ▶ We call the set of geometric graphs with given topology, edge lengths and incidence orders an **abstract geometric graph** \mathcal{G} .
 - ▶ A polygon P is **suitable** for \mathcal{G} if $\mathcal{S}(P) \in \mathcal{G}$.
 - ▶ A \mathcal{G} is **feasible** if it has a suitable polygon.
-
- ▶ Which \mathcal{G} are feasible?
 - ▶ If \mathcal{G} is feasible, are the suitable polygons unique?
 - ▶ How to construct suitable polygons?

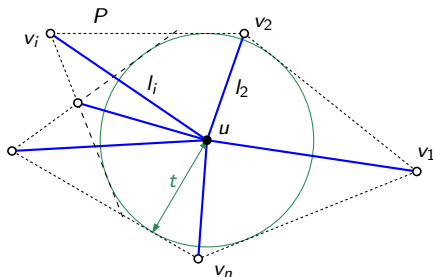
Star graphs S_n : introduction

- ▶ Let us start with simple trees: star graphs S_n .
 - ▶ A vertex u adjacent to n terminal vertices v_1, \dots, v_n .
 - ▶ We denote by l_i the length of uv_i . W.l.o.g. let $l_1 = \max_i l_i$.

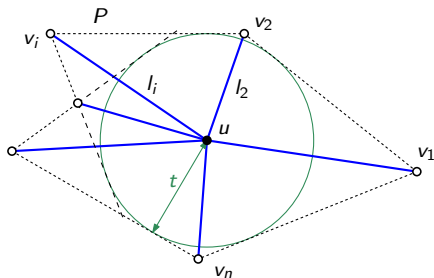


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 - ▶ A vertex u adjacent to n terminal vertices v_1, \dots, v_n .
 - ▶ We denote by l_i the length of uv_i . W.l.o.g. let $l_1 = \max_i l_i$.
- ▶ If P is suitable then u has equal orthogonal distance to all polygon edges.
 - ▶ Hence, there is a tangential circle with some radius t .



Star graphs S_n : introduction



Observation

If P is suitable for S_n then

1. two consecutive vertices cannot be both reflex,
2. $l_i < l_{i\pm 1}$ for a reflex v_i ,
3. the edges of P have equal orthogonal distance t to u , with $t \leq \min_i l_i$.

Constructing suitable polygons for S_n

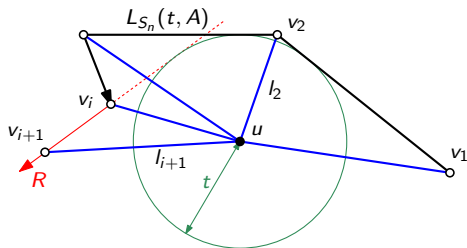
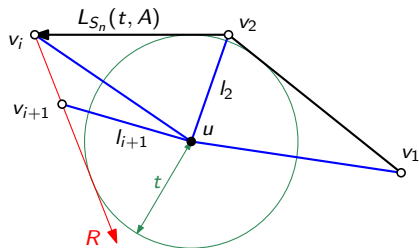
Is there a suitable polygon for S_n for a given convexity/reflexivity assignment A to its vertices?

Constructing suitable polygons for S_n

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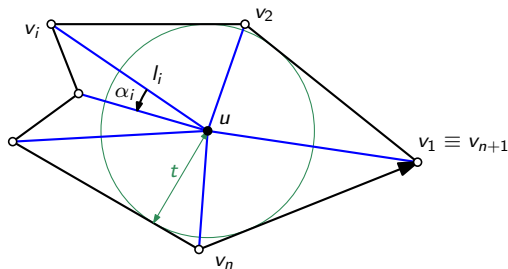
We construct the following polyline $L_{S_n}(t, A)$:

- ▶ Place a circle C with radius t and center $u = (0, 0)$ and a vertex v_1 at $(l_1, 0)$.
- ▶ We incrementally construct v_2, \dots, v_{n+1} :
 - ▶ Shoot a tangential ray R from v_i right to C .
 - ▶ Place v_{i+1} on the ray at desired distance $l_{1+i \bmod n}$ to u .



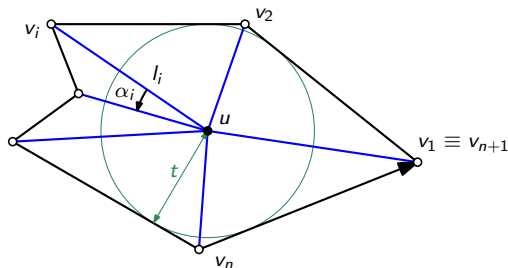
Constructing suitable polygons for S_n

- ▶ **Basic idea:** If $L_{S_n}(t, A)$ is closed and simple then $L_{S_n}(t, A)$ forms a suitable polygon.
- ▶ $L_{S_n}(t, A)$ is closed and simple if and only if $\alpha_A(t) := \sum_{i=1}^n \alpha_i = 2\pi$.



Constructing suitable polygons for S_n

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Lemma

$$\alpha_A(t) = 2 \sum_{\substack{i=1 \\ v_i \text{ convex}}}^n \arccos \frac{t}{l_i} - 2 \sum_{\substack{i=1 \\ v_i \text{ reflex}}}^n \arccos \frac{t}{l_i}. \quad (1)$$

Is a star graph S_n feasible?

Lemma

A suitable convex polygon for a star graph S_n exists if and only if $\sum_i \arccos \frac{\min_i l_i}{l_i} \leq \pi$. If a suitable convex polygon exists then it is unique.

Proof idea: Show that a $t \in (0, \min_i l_i]$ exists with $\alpha_A(t) = 2\pi$.

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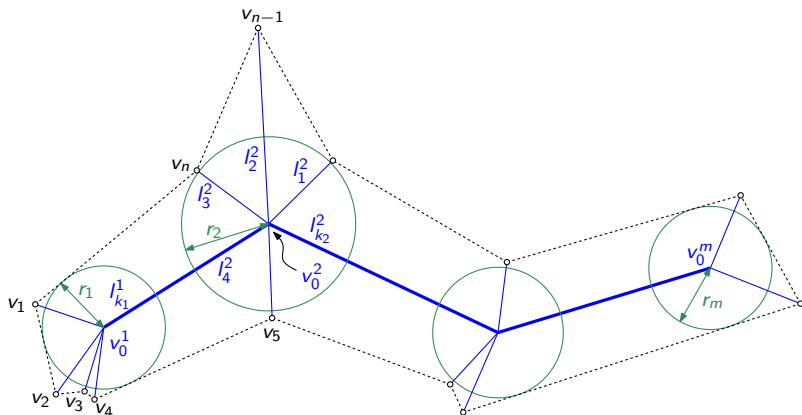
Proof idea: Show that a $t \in (0, \min_j l_j]$ exists with $\alpha_A(t) = 2\pi$.

Lemma

- ▶ *There exist infeasible star graphs S_5 .*
- ▶ *There exist star graphs S_5 for which multiple suitable polygons exist.*

Caterpillar graphs: notations

- ▶ A **caterpillar graph** G becomes a path (**backbone**) if all leaves are removed.
 - ▶ Backbone vertices are denoted by v_0^1, \dots, v_0^m .



Can we express the sum of inner angles of P as a function of one parameter?

Caterpillar graphs: geometric properties

Lemma

The radii r_2, \dots, r_m for some given caterpillar graph G are determined by r_1 according to the following recursions, for $1 \leq i < m$:

$$r_{i+1} = r_i + l_{k_i}^i \sin \beta_i \quad (2)$$

$$\beta_i = \beta_{i-1} + (1 - k_i/2)\pi + \quad (3)$$

$$\sum_{\substack{j=1 \\ v_j^i \neq v_0^{i-1}}}^{k_i-1} \begin{cases} \arcsin \frac{r_i}{l_j^i} & \text{if } v_j^i \text{ is convex} \\ \pi - \arcsin \frac{r_i}{l_j^i} & \text{if } v_j^i \text{ is reflex} \end{cases}$$

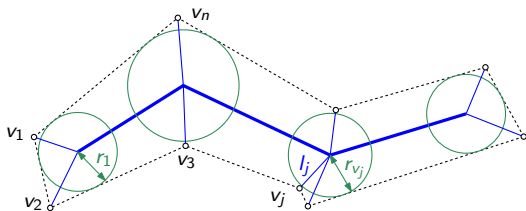
For $i = 1$ we define that $\beta_0 = 0$ and $v_j^1 \neq v_0^0$ being true for all $1 \leq j < k_1$.

Caterpillar graphs: feasibility and suitable polygons

Corollary

The sum of the inner angles of P with convexity assignment A is a function

$$\alpha_A(r_1) = 2 \sum_{j=1}^n \begin{cases} \arcsin \frac{r_{v_j}}{l_j} & \text{if } v_j \text{ is convex} \\ \pi - \arcsin \frac{r_{v_j}}{l_j} & \text{if } v_j \text{ is reflex} \end{cases} . \quad (4)$$



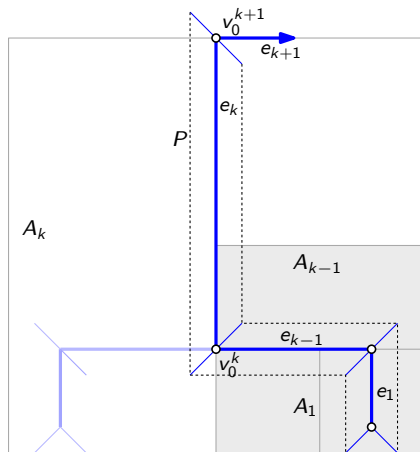
Lemma

There is only a finite number of suitable polygons for a caterpillar graph.

How many suitable polygons can exist?

Lemma

There exists a caterpillar graph with $3m$ vertices having 2^{m-2} suitable polygons.



Finish

Finish

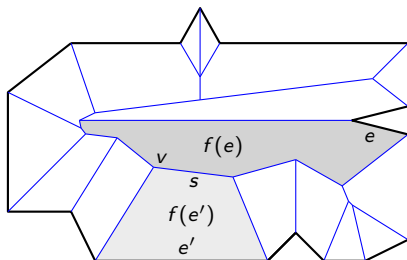
Bibliography



Aichholzer, O., Albers, D., Aurenhammer, F., and Gärtner, B. (1995).
A novel type of skeleton for polygons.
J. Universal Comp. Sci., 1(12):752–761.

Straight skeletons: basic geometric properties

- ▶ P is tessellated into faces.
 - ▶ Each face $f(e)$ belongs to an edge e .
- ▶ Every straight-skeleton edge s is on the boundary of two faces, $f(e)$ and $f(e')$, and lies on the bisector of e and e' .
- ▶ A straight-skeleton vertex v on the boundary of faces $f(e_1), \dots, f(e_k)$ has equal orthogonal distance to e_1, \dots, e_k .



Infeasible star graph S_5

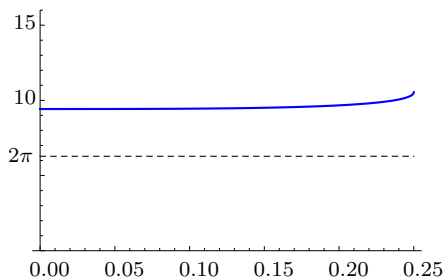
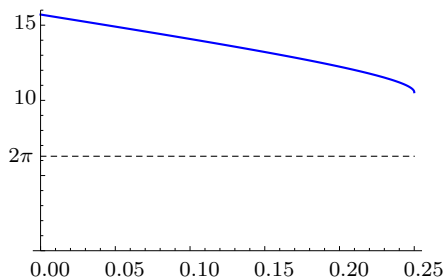
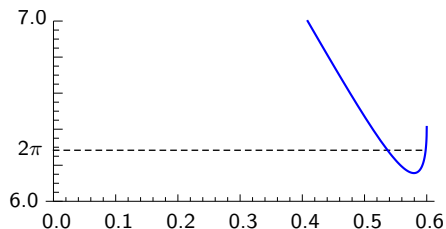


Figure : The sum $\sum_i \alpha_i$ for all $t \in (0, \min_i l_i]$, where $l_1 = \dots = l_4 = 1$ and $l_5 = 0.25$.
Left: v_5 is convex. Right: v_5 is reflex.

Multiple feasible polygons for S_5



$t = 0.5$

$t = 0.537$

$t = 0.57$

$t = 0.598$

$t = 0.6$

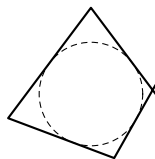
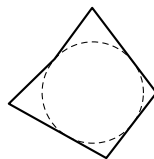
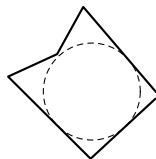
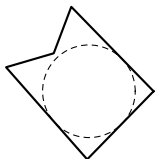
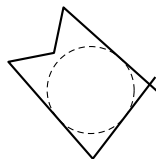


Figure : Edge lengths $l_1 = 0.75, l_2 = 1, l_3 = 0.6, l_4 = 1, l_5 = 0.79$. All vertices are convex, except for v_3 . Top: $\sum_i \alpha_i$ evaluates to 2π for two different values of t . Bottom: The result of our construction scheme for a sequence of different values of t .