Straight Skeletons
By Means of Voronoi Diagrams
Under Polyhedral Distance Functions

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Straight skeletons

- Wavefront propagation:
  - At time $t$ the wavefront $W_S(t)$ forms a mitered offset.
  - Events: structural changes of the wavefront over time.
- $S(P)$ is the set of loci traced out by vertices of $W_S(t)$. 
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Voronoi diagrams

- **Given:**
  - A normed space \((\mathbb{R}^d, \|\cdot\|)\).
  - A finite set \(S = \{s_1, \ldots, s_n\}\) of *input sites*.
- **Voronoi region** \(\mathcal{R}(s_i, S) = \{q \in \mathbb{R}^d : \|q - s_i\| \leq \|q - s_j\|, 1 \leq j \leq n\}\).
- **Voronoi diagram** \(\mathcal{V}(S) = \bigcup_{i=1}^{n} \partial \mathcal{R}(s_i, S)\).
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Voronoi diagram of a polygon

- Given: A polygon (with holes) $P$.
- Interpret the vertices and edges of $P$ as input sites $S$.
- $\mathcal{V}(P) = \mathcal{V}(S) \cap P$.

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\[ \mathcal{V}(P) \text{ tessellates } P \text{ into Voronoi regions.} \]
Straight skeleton versus Voronoi diagram

- The straight skeleton does not fit into the Abstract Voronoi Diagram framework of Klein.
- Computing $S(P)$ is $\mathcal{P}$-complete.
- The straight skeleton is prone to non-local effects.
- $S(P)$ changes discontinuously when moving vertices of $P$.

TL’DR: The straight skeleton is fundamentally different from the Voronoi diagram.
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The straight skeleton is prone to non-local effects.

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**TL’DR:** The straight skeleton is fundamentally different from the Voronoi diagram.

On the other hand:

- $P$ rectilinear, $(\mathbb{R}^2, \| \cdot \|_\infty)$: $\mathcal{V}(P) = S(P)$.
- $P$’s reflex vertices “rounded”, $(\mathbb{R}^2, \| \cdot \|_2)$: $\mathcal{V}(P) = S(P)$.

**Question**

Under which circumstances is $\mathcal{V}(P) = S(P)$?
Why?

Best of both worlds:

- Optimal algorithms for $\mathcal{V}(P)$ in $\mathbb{R}^2$ known, but not for $S(P)$.
- Definition for $S(P)$ in $\mathbb{R}^3$ is a pain, but not for $\mathcal{V}(P)$.
- $S(P)$ comprises piecewise-linear features only, but $\mathcal{V}(P)$ does not.
- $\mathcal{V}(P)$ changes continuously, $S(P)$ does not, et cetera.
Voronoi diagrams by means of wavefronts

- $X, Y \subseteq \mathbb{R}^d$:
  - $X \oplus Y = \{x + y : x \in X, y \in Y\}$.
  - $X \ominus Y = \{z \in \mathbb{R}^d : \{z\} \oplus Y \subseteq X\}$.
- Unit ball $B = \{x \in \mathbb{R}^d : \|x\| \leq 1\}$.
- Minkowski offset $W_V(t) = \partial(P \ominus tB)$.
\[ \mathcal{W}_v(t) = \partial(P \ominus tB) \]
\[ = P \cap \partial(\partial P \oplus tB) \]
\[ = P \cap \bigcup_{\text{face } s \text{ of } \partial P} \mathcal{R}(s, P) \cap \partial(s \oplus t \cdot B) \]
\( \mathcal{V}(P) \) is the interference pattern of the wavefront \( \mathcal{W}_V \).

The norm \( \| \cdot \| \) can be specified by a unit ball \( B \):

\[
\| x \|_B = \inf \{ t \geq 0 : x \in tB \} \quad \text{for any} \ x \in \mathbb{R}^d.
\]

**Question**

For which unit balls \( B \) and for which input shapes \( P \) is \( \mathcal{W}_S(t) = \mathcal{W}_V(t) \) for all \( t \geq 0 \)?
Proper unit balls

$B$ shall to be convex and $o$-symmetric.

$\mathcal{W}_S(t)$ has a piecewise-linear geometry.
- $\partial (P \ominus tB)$ comprises features of $P$ and $B$.
- For $\mathcal{W}_S(t) = \mathcal{W}_V(t)$, $B$ needs to be polyhedral.

At least for $P = B$ we would like that $\mathcal{W}_S(t) = \mathcal{W}_V(t)$.
- $\mathcal{W}_V(t) = (1 - t)B$.
- All facets of $\mathcal{W}_V$ reach $o$ at time 1.
- All facets of $\mathcal{W}_S$ need to reach $o$ at time 1.
- All facets of $B$ have distance 1 to $o$.
- We call such a $B$ isotropic.
Proper unit balls

**Definition**

A proper unit ball is a convex, $o$-symmetric, isotropic polyhedron.

**Lemma**

For a proper unit ball $B$ and any $v \in \mathbb{R}^d$ it holds that $\|v\|_2 \geq \|v\|_B$, and equality holds exactly when $v$ is a normal vector of a facet of $B$. 
Proper input shapes

**Definition**

A \((d\text{-dimensional})\) input shape \(P\) is a connected, compact set in \(\mathbb{R}^d\) whose boundary forms a polyhedral surface that constitutes an orientable \((d - 1)\)-manifold.

**Definition**

A face \(f\) of \(P\) of dimension at most \(d - 2\) is called reflex if for any point \(p\) in the relative interior of \(f\) and for any small enough Euclidean ball \(O\), centered at \(p\), \(O \setminus P\) is contained in a half-space whose boundary supports \(p\).
Corresponding facets

For a facet \( f \) of \( P \) let \( n(f) \) be the normal vector of \( f \) pointing to the interior.

Lemma

Every facet \( f \) of \( P \) has a corresponding facet \( f^B \) of \( B \) that has \( n(f) \) as the outer normal vector, unless \( \mathcal{W}_V(\varepsilon) \neq \mathcal{W}_S(\varepsilon) \) for some \( \varepsilon > 0 \).
Two-dimensional input shapes

The last lemma says:

- For every edge $e$ of $P$ there is a corresponding edge $e^B$ of $B$.

**Lemma**

Let $v$ be a reflex vertex of $P$ with incident edges $e_1$ and $e_2$. Then there is a corresponding vertex $v^B$ of $B$ that is incident to $e_1^B$ and $e_2^B$, unless $W_V(\varepsilon) \neq W_S(\varepsilon)$ for some $\varepsilon > 0$.

The existence of corresponding edges and reflex vertices is necessary for $W_V(t) = W_S(t)$.
Two-dimensional input shapes

**Definition**

A *proper* input shape $P$ w.r.t. a proper unit ball $B$ in $\mathbb{R}^2$ is a polygon with holes such that

1. for each edge $e$ of $P$ there is a corresponding edge $e^B$ of $B$ whose outer normal vector is $n(e)$ and
2. for each reflex vertex $v$ of $P$, incident to edges $e_1$ and $e_2$, there is a corresponding vertex $v^B$ of $B$ that is incident to $e_1^B$ and $e_2^B$.

**Theorem**

For a proper input shape $P$ w.r.t. a proper unit ball $B$ in $\mathbb{R}^2$ it holds that $W_S(t) = W_V(t)$ for all $t \geq 0$. 
Higher-dimensional input shapes

We know: each facet $f$ of $P$ has a corresponding facet $f^B$ in $B$.

For $d = 2$: a proper input shape looks locally the same as a unit ball at non-convex features.
  - For $d > 2$ we have a larger “diversity” of non-convexity.
  - For $(d - 2)$-dimensional faces the situation is still simpler.

Lemma

Let $P$ be an input shape in $\mathbb{R}^d$, where $d \geq 2$. For each reflex $(d - 2)$-dimensional face $e$ of $P$, which is incident to facets $f_1$ and $f_2$, it holds that $f_1^B \cap f_2^B \neq \emptyset$, unless $\mathcal{W}_V(\varepsilon) \neq \mathcal{W}_S(\varepsilon)$ for some $\varepsilon > 0$. 
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From $f_1^B \cap f_2^B \neq \emptyset$ it does not follow that $f_1^B \cap f_2^B$ forms a $(d - 2)$-dimensional face of $B$!
Proper input shapes

Definition

An input shape $P$ in $\mathbb{R}^d$ is called proper w.r.t. a proper unit ball $B$ if

(I1) for each facet $f$ of $P$ there is a corresponding facet $f^B$ of $B$ whose outer normal vector is $n(f)$ and

(I2) for all points $p$ on all facets $f$ of $P$, there is a point $p'$ such that
$$\inf_{q \in P} \|p' - q\|_B = \|p' - p\|_B > 0 \text{ and } p \in \text{relint}_f (f \cap (p' + \|p' - p\|_B \partial B)),$$

Lemma

For any proper input shape $P$ w.r.t. $B$ there is a finite point set $S$, with $P \cap S = \emptyset$, and some $\epsilon > 0$ such that $\partial P \subseteq \partial (S \oplus \epsilon B)$. 

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**Lemma**

For any proper input shape $P$ w.r.t. $B$ there is a finite point set $S$, with $P \cap S = \emptyset$, and some $\varepsilon > 0$ such that $\partial P \subseteq \partial (S \oplus \varepsilon B)$. 
Corresponding reflex faces

**Lemma**

Let $e$ be a reflex face of dimension $k$ of a proper input shape $P$ in $\mathbb{R}^d$, where $0 \leq k \leq d - 2$. Then for any point $p \in \text{relint } e$ there is a point $p^B \in \partial B$ such that for some $\epsilon, \epsilon' > 0$ the sets $\partial P \cap (p + \epsilon O)$ and $\partial B \cap (p^B + \epsilon' O)$ are homothetic, where $O$ denotes the Euclidean unit ball. In particular, to $e$ corresponds a $k$-dimensional face $e^B$ of $B$ with $p^B \in \text{relint } e^B$.

**Theorem**

For a proper input shape $P$ w.r.t. a proper unit ball $B$ in $\mathbb{R}^d$ it holds that $W_S(t) = W_Y(t)$ for all $t \geq 0$. 
Approximation by proper input shapes

Lemma

For any input shape \( P \) and any \( \varepsilon > 0 \) there is proper input shape \( P' \) with \( P \subseteq P' \subseteq P \oplus \varepsilon O \).

Basically, the set of proper input shapes lies dense in the set of input shapes.
Thank you for your attention

Questions?