

Realizability of Planar Straight-Line Graphs as Straight Skeletons

Stefan Huber

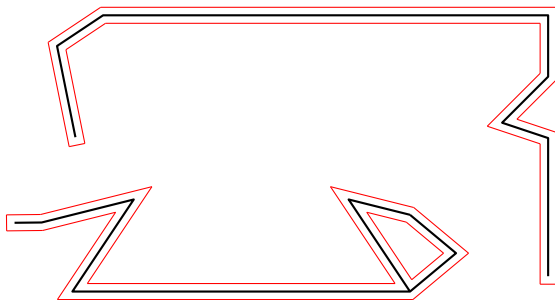
Joint work with Therese Biedl and Martin Held.

Institute of Science and Technology Austria

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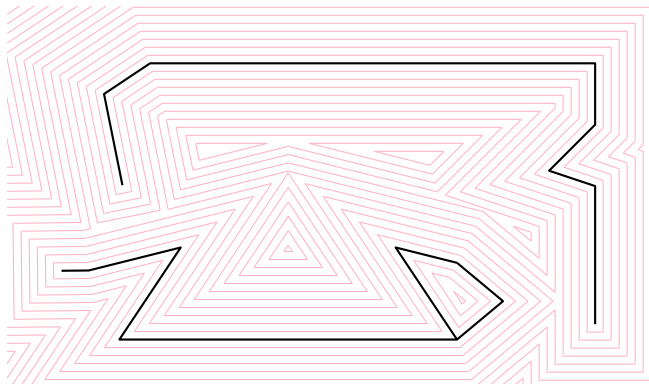
Straight skeleton of a PSLG

- ▶ [Aichholzer and Aurenhammer, 1998]: straight skeleton $\mathcal{S}(G)$ of a PSLG G



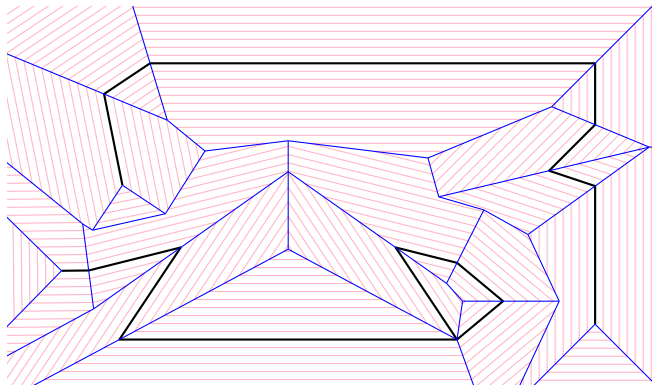
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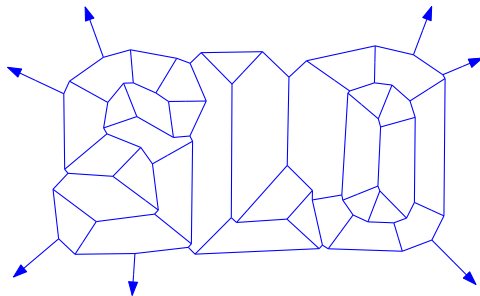


Problem statement

PSLG[∞]: edges may be straight-line segments or rays.

Problem (GMP-SS)

Given a PSLG[∞] G , can we find a PSLG H such that $\mathcal{S}(H) = G$?

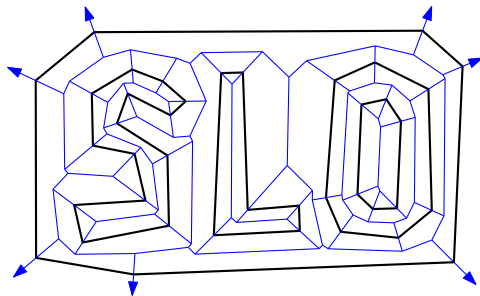


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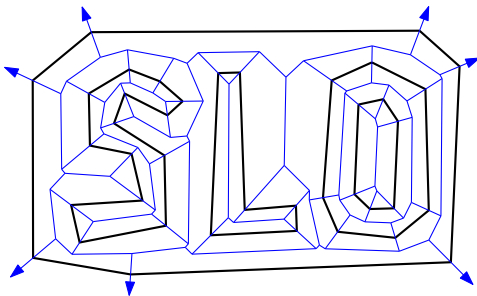


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Problem [Aichholzer et al., 1995]

Give necessary and sufficient conditions for G to be the straight skeleton of H .

Prior work

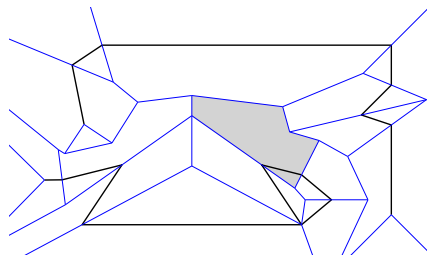
[Aichholzer et al., 2012]:

- ▶ Any **abstract tree** T can be realized as $\mathcal{S}(P)$ (or $\mathcal{V}(P)$) of a convex polygon.
- ▶ Realizability of **phylogenetic trees** T as $\mathcal{S}(P)$ of a polygon P .

Characterization: basic facts

Facts

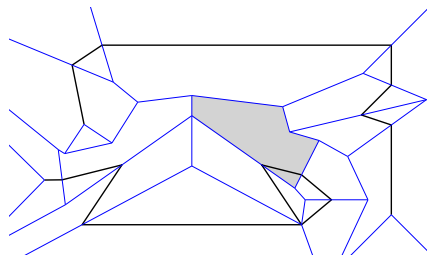
- ▶ Every edge of $\mathcal{S}(H)$ is on the bisector of two edges of H .
- ▶ Every face of $\mathcal{S}(H)$ contains exactly one segment of H , except for faces generated by degree-one vertices of H .
- ▶ Every edge of H begins and ends at an edge of $\mathcal{S}(H)$.
- ▶ If a vertex of $\mathcal{S}(H)$ has degree two then it coincides with a degree-one vertex of H . All other vertices have degree three or higher.



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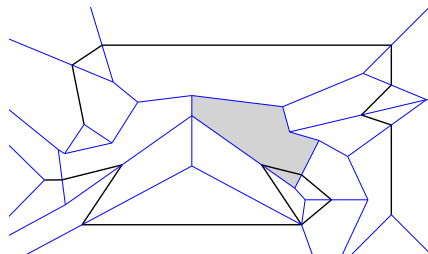
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Temporary assumption: G has no degree-2 vertices.

Characterization: inside-condition

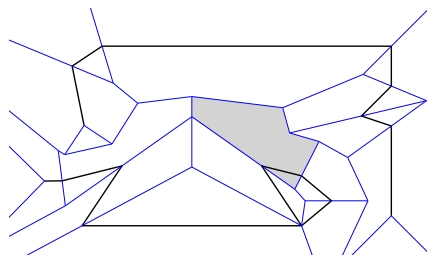
Let G be the putative straight skeleton and F the set of faces of G .



A solution to GMP-SS can be denoted as a mapping $\lambda: F \rightarrow \mathcal{L}$, where \mathcal{L} is the set of lines.

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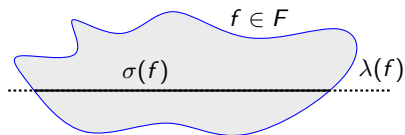
A solution to GMP-SS can be denoted as a mapping $\lambda: F \rightarrow \mathcal{L}$, where \mathcal{L} is the set of lines.

Definition (Inside-condition)

λ fulfills the inside-condition if $\sigma(f) := \lambda(f) \cap f$ is a **single line segment** for all $f \in F$.

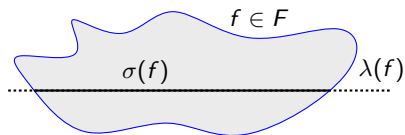
Characterization: inside-condition

We construct H as the graph whose edges are $\sigma(f)$, with $f \in F$.



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For a G and λ we denote by $G^* := G \cup H$ and by F^* the faces of G^* .

- ▶ Every face of G contains two faces of G^* .
- ▶ We reuse $\lambda(f)$ and $\sigma(f)$ for faces of G^* accordingly.

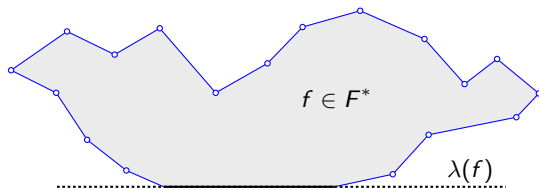
Characterization: sweeping-condition

Definition (Sweeping-condition)

A face f of G^* fulfills the sweeping-condition if

1. f is **monotone** w.r.t. $\lambda(f)$ and
2. at the lower chain, the **distance to $\lambda(f)$ is increasing**, when moving away from $\sigma(f)$.

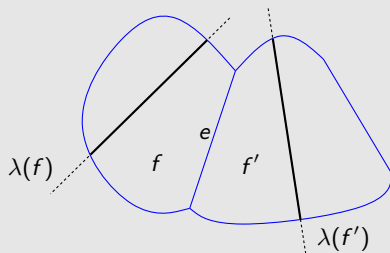
λ fulfills the sweeping-condition if all faces of G^* fulfill it.



Characterization: bisector-condition

Definition (Bisector-condition)

The edge $e = f \cap f'$ fulfills the bisector-condition if e lies on the bisector of $\lambda(f)$ and $\lambda(f')$.



λ fulfills the bisector-condition if all edges of G fulfill the bisector-condition.

Characterization

Lemma

If λ solves GMP-SS then λ fulfills the inside-, sweeping-, and bisector-condition.

Proof. Inside- and bisector-condition: by definition of straight skeletons.

Sweeping-condition:

- ▶ Monotonicity by [Aichholzer et al., 1995].
- ▶ Lower chain is even convex by [Huber, 2012].



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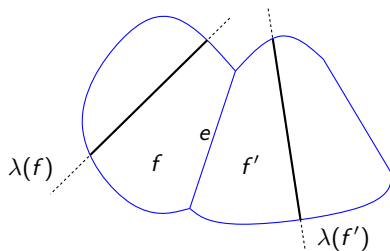
Theorem

If λ fulfills the inside-, sweeping-, and bisector-condition then λ solves GMP-SS.

Recognizing straight skeletons

Key method: We successively reflect lines $\lambda(f)$ at edges of f .

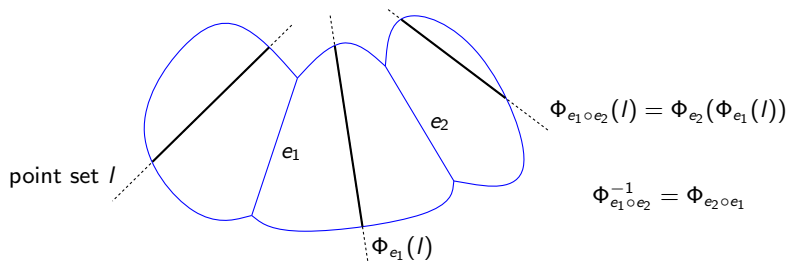
- ▶ Assume we know a suitable $\lambda(f)$ for one face f .
- ▶ Bisector-condition: we know $\lambda(f')$ for a neighboring face f' , too.
- ▶ Going along a spanning tree of the dual of G , we know $\lambda(f')$ for all $f' \in F$.



Recognizing straight skeletons

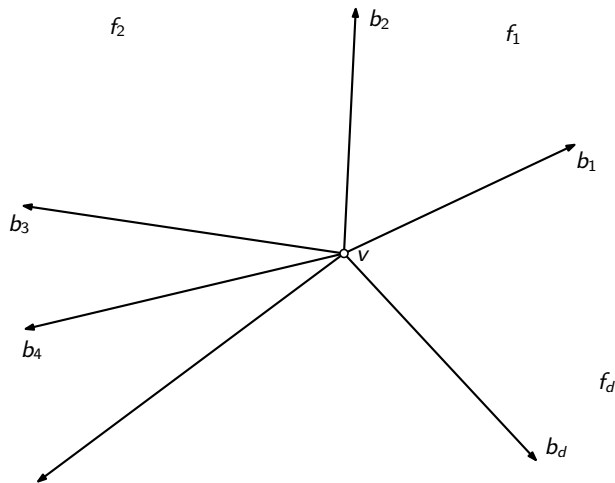
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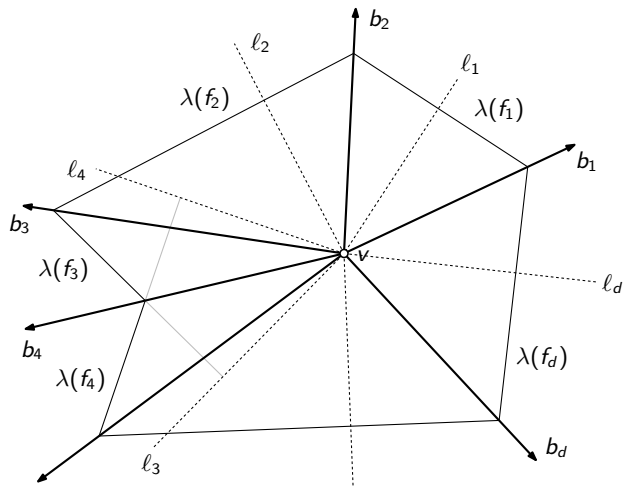
Recognizing straight skeletons: star graphs

- ▶ “Local view” at a vertex v of G with incident ray-edges b_1, \dots, b_d .
 - ▶ Find λ that fulfills inside-, (sweeping-), and bisector-condition.



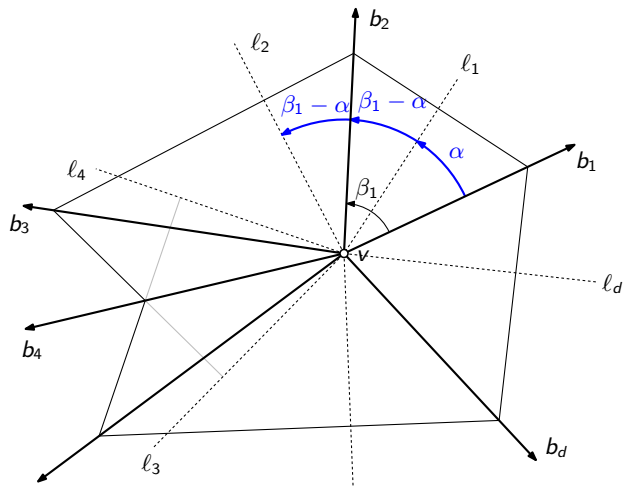
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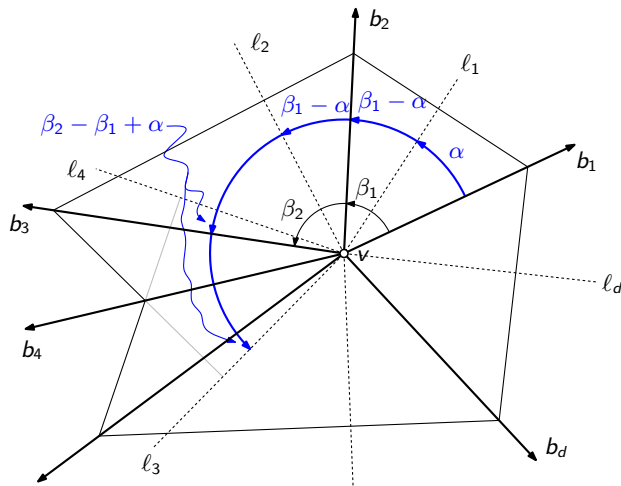
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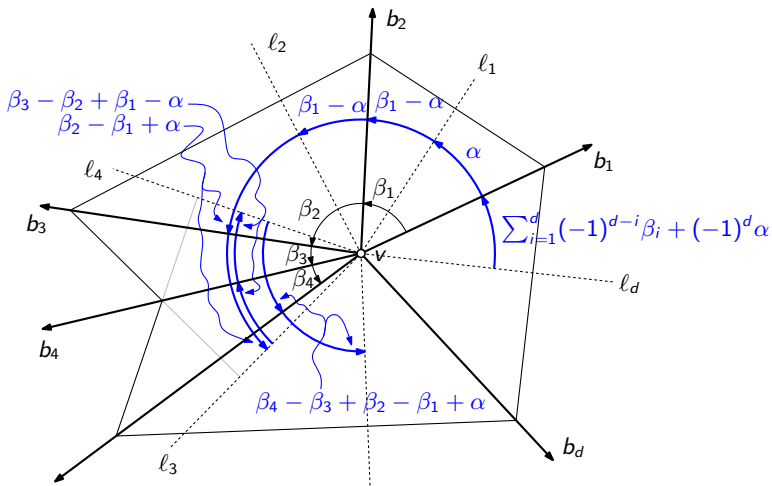
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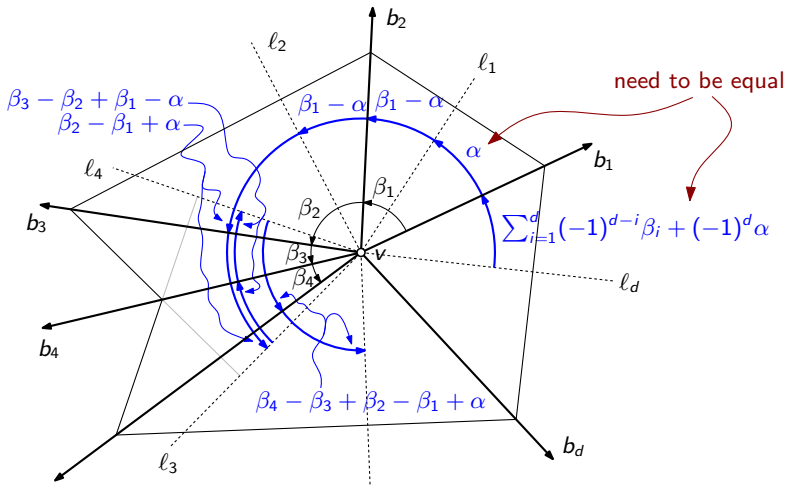
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Recognizing straight skeletons: star graphs

We get $\alpha = \sum_{i=1}^d (-1)^{d-i} \beta_i + (-1)^d \alpha$ and therefore

$$\frac{1}{2} \sum_{i=1}^d (-1)^{d-i} \beta_i = \begin{cases} 0 & \text{if } d \text{ is even,} \\ \alpha & \text{if } d \text{ is odd.} \end{cases} \quad (1)$$

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Definition

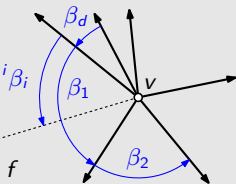
The vertex v with even degree d fulfills the **balance-condition** if

$$\beta_d - \beta_{d-1} + \dots + \beta_2 - \beta_1 = 0.$$

For vertices of odd degree d we define $\ell(f, v)$ as

$$\frac{1}{2} \sum_{i=1}^d (-1)^{d-i} \beta_i$$

$$\ell(f, v) :=$$

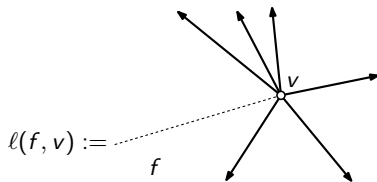


Recognizing straight skeletons: star graphs

Lemma

$\Phi_{b_1 \circ \dots \circ b_d}(l) = l$ if and only if

$$\begin{cases} v \text{ fulfills the balance-condition} & \text{if } d \text{ is even} \\ l = \ell(f, v) \vee l \perp \ell(f, v) \text{ for some } f \in F \text{ with } v \in f & \text{if } d \text{ is odd} \end{cases} \quad (2)$$



Recognizing straight skeletons: PSLGs

The previous lemma imposes constraints on λ for the vertices of G :

$$\ell(f) := \{l \in \mathcal{L} : l \cap \text{int } f \neq \emptyset\} \cap \bigcap_{\substack{v \text{ is vertex of } f \\ \deg(v) \text{ is odd}}} \{\ell(f, v)\} \cup \ell(f, v)^\perp. \quad (3)$$

We propagate the per-face constraints to a single face:

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We propagate the per-face constraints to a single face:

- ▶ Choose a spanning tree T of the dual of G , with a root face r .
- ▶ Denote by $f \rightsquigarrow^T r$ the sequence of edges in T from f to r and define

$$f^r := \Phi_{f \rightsquigarrow^T r}(f) \quad (4)$$

$$\ell^r(f) := \Phi_{f \rightsquigarrow^T r}(\ell(f)) \quad (5)$$

$$X := \bigcap_{f \in F} \ell^r(f). \quad (6)$$

Recognizing straight skeletons: PSLGs

Theorem

GMP-SS for G has a solution if and only if

- ▶ *the balance-condition holds for all vertices of even degree and*
- ▶ *there is a line $l \in X$ such that for all $f \in F$*
 - ▶ *$l \cap f^r$ is a single segment and*
 - ▶ *the components of $f^r \setminus l$ fulfill the sweeping-condition.*

There is a one-to-one correspondence between such lines $l \in X$ and solutions to GMP-SS.

Recognizing straight skeletons: PSLGs

Proof sketch:

- ▶ Take a suitable I and define $\lambda(f) := \Phi_{r \rightsquigarrow T_f}(I)$.
- ▶ To show: λ fulfills the inside-, bisector- and sweeping-condition.
 - ▶ Inside- and sweeping-condition are fulfilled by assumption.
 - ▶ Bisector-condition for (duals of) edges in T as well.

Recognizing straight skeletons: PSLGs

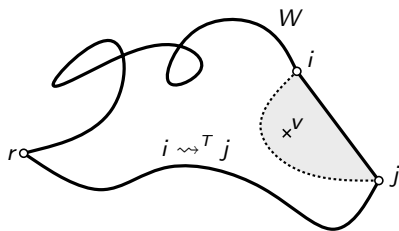
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- ▶ **Claim:** edges not in T fulfill the bisector-condition as well.
- ▶ **Stronger claim:** Let W be any walk in the dual of G from r to j . Then $\Phi_W(\lambda(r)) = \lambda(j)$. That is, it does not matter how we choose T .



Reconstructing the input: algorithm

We are given G and want to find a suitable λ , i.e., a suitable $l \in X$.

- ▶ Check that balance-condition holds at every even-degree vertex.
- ▶ We compute T , all $f^r = \Phi_{f \rightsquigarrow T_r}(f)$ and all $l^r(f, v) = \Phi_{f \rightsquigarrow T_r}(l(f, r))$ in total linear time.

Reconstructing the input: algorithm

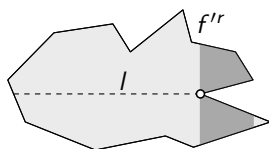
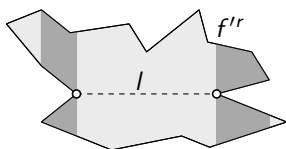
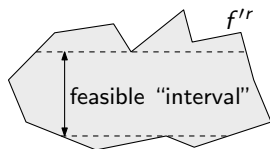
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- ▶ **Case 1:** All vertices have even degree.
 - ▶ By the balance-condition all faces are convex.
 - ▶ Sweeping-condition is trivial.
 - ▶ Using [Edelsbrunner et al., 1989] and [Hershberger, 1989] we find all lines l traversing all int f^r in $O(n \log n)$ time.

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- ▶ **Case 2:** At least one vertex v has odd degree.
 - ▶ A suitable l has fixed direction: identical or perpendicular to $\ell^r(f, v)$.
 - ▶ inside/sweeping condition \Rightarrow restrict all suitable l to an “interval” of parallel lines in $O(n)$ time.



Reconstructing the input

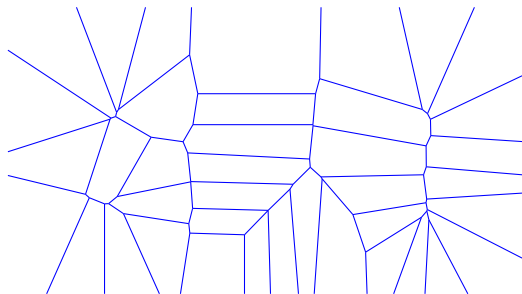
Theorem

GMP-SS can be solved and the set of solutions can be found in $O(n \log n)$ time of a $PSLG^\infty$ G with n edges.

Voronoi diagrams

Problem (GMP-VD)

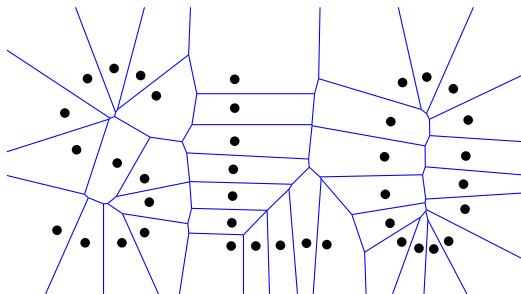
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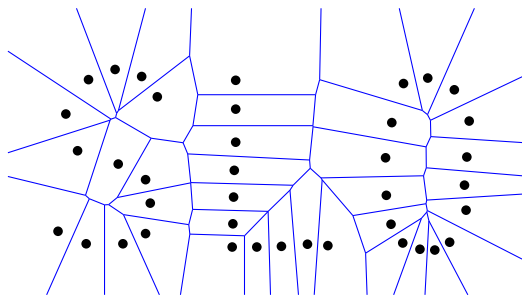
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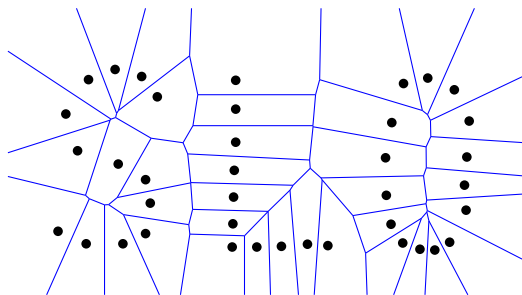
Prior work:

- ▶ [Ash and Bolker, 1985]: Solve GMP-VD if all vertices have **odd degree**.

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Prior work:

- ▶ [Ash and Bolker, 1985]: Solve GMP-VD if all vertices have **odd degree**.
- ▶ [Hartvigsen, 1992]: Solve GMP-VD by means of linear programming.

Characterization of Voronoi diagrams

We denote a solution of GMP-VD as a mapping $\rho: F \rightarrow \mathbb{R}^2$.

- ▶ We look for ρ such that $\mathcal{V}(\{\rho(f): f \in F\}) = G$.

Lemma ([Ash and Bolker, 1985])

ρ solves GMP-VD if

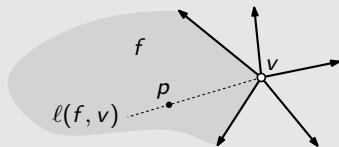
- ▶ **Inside-condition:** $\rho(f) \in \text{int } f$ for all $f \in F$.
- ▶ **Bisector-condition:** e is on the bisector of $\rho(f), \rho(f')$ for any edge $e = f \cap f'$.

Recognizing Voronoi diagrams

Lemma

$\Phi_{b_1 \circ \dots \circ b_d}(p) = p$ if and only if

$$\begin{cases} v \text{ fulfills the balance-condition} & \text{if } d \text{ is even} \\ p \in \ell(f, v) \text{ for some } f \in F \text{ with } v \in f & \text{if } d \text{ is odd} \end{cases} \quad (7)$$

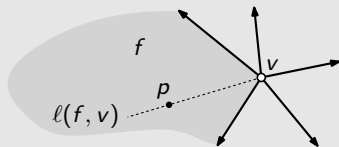


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We again define

$$S(f) := (\text{int } f) \cap \bigcap_{\substack{v \text{ is vertex of } f \\ \deg(v) \text{ is odd}}} \ell(f, v) \quad \text{Easily becomes a single point.} \quad (8)$$

$$X := \bigcap_{f \in F} \Phi_{f \rightsquigarrow T_r}(S(f)) \quad \text{Every point implies a solution } \rho. \quad (9)$$

Conclusion

Characterization of straight skeletons:

- ▶ Deeper insight in the geometry and structure of $\mathcal{S}(H)$.
- ▶ Allows for necessary and sufficient $O(n)$ time a-posteriori checks of the validity of $\mathcal{S}(H)$ in straight-skeleton codes.

We solve GMP-SS and GMP-VD on G

- ▶ using a **unified framework** based on reflections on edges of a spanning tree of the dual of G
- ▶ in $O(n \log n)$ time.
- ▶ First result on GMP-SS.
- ▶ Closes a gap in [Ash and Bolker, 1985] for GMP-VD when vertices have even degree.

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Characterization: Proof

Key idea: G and $\mathcal{S}(H)$ each impose a wavefront-propagation process, $\mathcal{W}_G(t)$ and $\mathcal{W}_{\mathcal{S}(H)}(t)$.

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Lemma

The initial wavefronts $\mathcal{W}_G(\epsilon)$ and $\mathcal{W}_{\mathcal{S}(H)}(\epsilon)$ are identical.

Characterization: Proof

Lemma

Assume that $\mathcal{W}_G(t') = \mathcal{W}_{S(H)}(t')$ for $0 < t' < t$.

- ▶ If $\mathcal{W}_G(t)$ hits a vertex v of G^* , then v coincides with a vertex of $S(H)$.
- ▶ Analogously for $\mathcal{W}_{S(H)}$.

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Theorem

$\mathcal{W}_G(t) = \mathcal{W}_{S(H)}(t)$ for all t .

Proof. [Sketch]

- ▶ By induction on the chronological order when \mathcal{W}_G resp. $\mathcal{W}_{S(H)}$ hits a vertex v of G resp. $S(H)$.
- ▶ In a neighborhood of v we have swept and not-yet-swept cones.
- ▶ Insight: In the not-yet-swept cones contain each exactly one “outgoing” edge of G resp. $S(H)$.
- ▶ Claim: these edges are identical in the neighborhood of v .

Non-unique solutions to GMP-SS and GMP-VD

