Topological Considerations for the Incremental Computation of Voronoi Diagrams of Straight-Line Segments and Circular Arcs

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Problem

Devise and implement an algorithm for computing the Voronoi diagram of points, straight-line segments and circular arcs for real-world applications.

Idea

Extend the incremental algorithm of (Imai, 1996) resp. (Held, 2001), which handles points and straight-lines, to circular arcs.

In this talk, we will pick a few topological and graph-theoretical aspects when incrementally constructing Voronoi diagrams.
Basic definitions: Voronoi diagram

### Definition (proper input set)

A finite disjoint system $S \subseteq \mathcal{P}(\mathbb{R}^2)$ is called *proper* set of input sites, if

- $S$ consists of points, open segments and open arcs (less than semi-circles),
- $S$ contains the endpoints of the segments and arcs as well.

Figure: Cone of influence $CI(s)$, of a point, segment or arc $s$. 
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Topological Constraints of VDs of Segments and Arcs
Voronoi diagrams

Definition (Voronoi cell, polygon, diagram)

Let $S$ be a proper set of input sites, $s \in S$ an input site and $d$ be the Euclidean distance. We define the Voronoi cell of $s$ as

$$\mathcal{VC}(s, S) := \text{cl}\{p \in \text{int} \mathcal{CI}(s) : d(p, s) \leq d(p, S \setminus \{s\})\}.$$ 

The Voronoi polygon $\mathcal{VP}(s, S)$ is commonly defined as the boundary of $\mathcal{VC}(s, S)$ and the Voronoi diagram is defined as the union of all Voronoi polygons:

$$\mathcal{VD}(S) := \bigcup_{s \in S} \mathcal{VP}(s, S).$$
Voronoi diagrams

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$$\mathcal{VD}(S) := \bigcup_{s \in S} \mathcal{VP}(s, S).$$
Suppose we would define a Voronoi cell $\mathcal{VC}(s, S)$ as 
$$\{ p \in CI(s) : d(p, s) \leq d(p, S \setminus \{s\}) \}.$$ 

All points $p$ from the center of $s_1$ to the common endpoint of the tangential sites $s_1$ and $s_2$ would belong to $\mathcal{VC}(s_2, S)$ as well!
(Sugihara & Iri, 1992) presented a topology-oriented incremental algorithm for points.

(Imai, 1996) sketched an extension to segments.

(Held, 2001) filled missing algorithmic gaps and cast the algorithm into an implementation: VRONI.
Let $S^+ := S \cup \{s\}$ be a proper set of input sites, with an arc $s \notin S$. Suppose that we already know $VD(S)$ and we want to insert the arc $s$ into the Voronoi diagram $VD(S)$. 

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We consider the Voronoi cells of the endpoints of $s$: within each cell we mark the Voronoi node whose clearance disk is “violated most” by $s$. We call these two nodes *seed nodes*.
Basic incremental algorithm

Starting from a seed node, recursively mark further nodes if their clearance disk is intersected by $s$. 
Basic incremental algorithm

- Remove marked edges,
- compute new nodes and adapt semi-marked edges,
- connect new nodes with edges.
Question 1
Is there always an appropriate seed node?

Question 2
Is a marked edge always completely in the future Voronoi cell \( VC(s, S^+) \)?
Existence of seed node

Lemma

Let \( p \) be an endpoint of an arc \( s \). There always exists a node \( v \in \mathcal{VP}(p, S) \), with \( v \in \mathcal{CI}(s) \), hence \( v \in \mathcal{VC}(s, S^+) \).

Based on this lemma, we select a seed node as follows:

- If \( \exists v \in \text{int} \mathcal{CI}(s) \) then all nodes in \( \mathcal{VP}(p, S) \cap \mathcal{CI}(s) \) are connected by marked edges \( \Rightarrow \) we can choose any of them as seed node.
- Otherwise, we distinguish the following cases:
  1. The arc \( s \) meets exactly one site \( s' \) tangentially in \( p \).
  2. Several sites meet in a common endpoint \( p \).
Existence of seed node

Lemma

Let $p$ be an endpoint of an arc $s$. There always exists a node $v \in \mathcal{VP}(p, S)$, with $v \in CI(s)$, hence $v \in VC(s, S^+)$. 

Based on this lemma, we select a seed node as follows:

- If $\exists v \in \text{int}CI(s)$ then all nodes in $\mathcal{VP}(p, S) \cap CI(s)$ are connected by marked edges $\Rightarrow$ we can choose any of them as seed node.
- Otherwise, we distinguish the following cases:
  1. The arc $s$ meets exactly one site $s'$ tangentially in $p$.
  2. Several sites meet in a common endpoint $p$.
Tangential sites

- $s$ meets exactly one site $s' \in S$ tangentially in $p$.
- $e_1, e_2 \in \mathcal{VP}(p, S)$ originate from $p$,
- on a supporting line.
We do not mark nodes coinciding with input points. The other two nodes of $e_1, e_2$ are the only candidates. Based on case analysis: Select proper candidate.
Several sites $s_1, s_2 \ldots$ meet in $p$.

1. Scan the nodes $v \in \mathcal{V}_p(S)$ and check whether $v \in \mathcal{CI}(s)$ and check for non-zero clearance.

2. If no such node exists...
...scan edges $e_1, e_2, \ldots$ which are incident to $v_1, v_2, \ldots$.

Test whether clearance disk of a second node of $e_i$ is intersected by $s$ and choose such a node as seed node.

Again, take care of tangential sites.
Question 1
Is there always an appropriate seed node?

Question 2
Is a marked edge always completely in the future Voronoi cell $\mathcal{VC}(s, S^+)$?
Theorem

Let $G$ be the graph corresponding to the nodes and edges of $\mathcal{VD}(S)$ which completely lie in $\mathcal{VC}(s, S^+)$, but do not intersect with $(\text{cl } s) \setminus s$. Then $G$ forms a tree.
Let $T$ be the graph of marked edges. It can be shown that

- $T$ contains a cycle if and only if $T$ contains an edge which is not completely contained in $\mathcal{VC}(s, S^+)$, and,
- therefore, should be partly preserved.
Tree structure of marked edges

Question
Can it happen that $T$ contains an edge which should be partly preserved?

Answer: Actually, it can happen.
Solution: Search for cycles in $T$ and break them up.
There are known examples (Held, 2001), where a cycle of edges is marked.

Solution: Split every edge at its apex by a degree-2 node, if it contains the apex in its relative interior.

Note
In the sequel, we assume that no edge contains the apex in its relative interior. (Apex splitting)
It turns out that apex splitting is not sufficient when considering arcs. We distinguish the following cases:

1. An edge has been marked and reaches outside of $CI(s)$.
2. An edge has been marked which is completely in $CI(s)$, but partly remains in $VD(S^+)$. 

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Breaking up a cycle outside of $CI$

- Suppose that two arcs $s_1, s_2$ define a hyperbolic edge $e$, as illustrated.
- $v_1, v_2$ are the end nodes of $e$. 
Breaking up a cycle outside of $\mathcal{I}$

- $g$ is the supporting line of a secant of $e$.
- $V$ is the union of projection segments of $e$ on $s_1$ and $s_2$.
- we choose a point on each of the two parts of $g \setminus (V \cup CD(v_1, S) \cup CD(v_2, S))$,
- such that their normals onto $g$ intersect $CD(v_1, S)$ resp. $CD(v_2, S)$. 

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Now we can insert a semi-circle between the two chosen points as illustrated.

\( e \) is marked, but reaches outside of \( CI(s) \).
Breaking up a cycle outside of $\mathcal{CI}$

Solution: Let us denote with $p$ the intersection of $e$ and the line through the centers of $s_1$ and $s$. It can be shown that $p \notin \mathcal{VC}(s, S^+)$, hence $p$ is a proper split point.
Implementation

- extended Held’s VRONI to circular arcs
- ANSI C
- double-precision floating-point arithmetic
- tested on several hundred synthetic and real-world data sets
Randomized insertion results in $O(n \log n)$ expected runtime

Experimental evaluation “yields a close to linear” behaviour:
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Topological Constraints of VDs of Segments and Arcs
VRONI: An Engineering Approach to the Reliable and Efficient Computation of Voronoi Diagrams of Points and Line Segments. 
*Comput. geom. theory and appl.*, 18(2), 95–123.

Imai, T. 1996. 
A Topology Oriented Algorithm for the Voronoi Diagram of Polygons. 
Pages 107–112 of: *Proc. canad. conf. comput. geom. (cccg’96)*. 
Ottawa, Canada: Carleton University Press.
Breaking up a cycle inside of $\mathcal{CI}$

- $e$ is an edge completely contained in $\mathcal{CI}(s)$.
- $v_1$ and $v_2$ are marked, but $e$ contains points not in $\mathcal{VC}(s, S^+)$.
Breaking up a cycle inside of $CI$

Solution: Let us denote with $p$ the intersection of $e$ and the projection line of the center of $s$ on $s_1$. It can be shown that $p \notin \mathcal{VC}(s, S^+)$. Hence $p$ is a proper split point.