A Practice-Minded Approach to Computing Motorcycle Graphs

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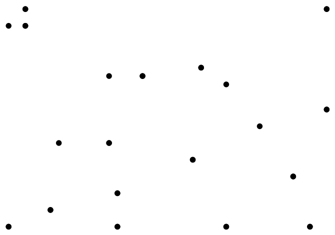
What is a motorcycle graph?

- We define a **motorcycle** m as a triple $(p, s, t^*) \in \mathbb{R}^2 \times \mathbb{R}^2 \times [0, \infty)$, where p is the start point, t^* is the start time and s is the speed vector.
- Consider n motorcycles $m_1, \ldots, m_n$, with $m_i = (p_i, s_i, t_i^*)$. Each motorcycle leaves a trace behind and crashes when reaching the trace of another motorcycle.



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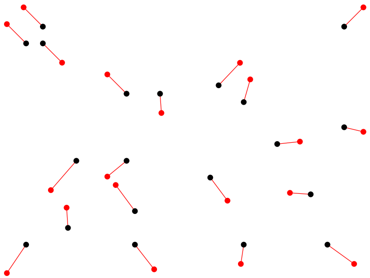

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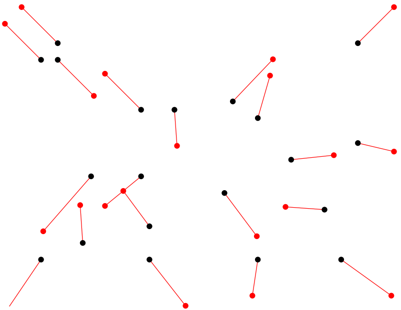

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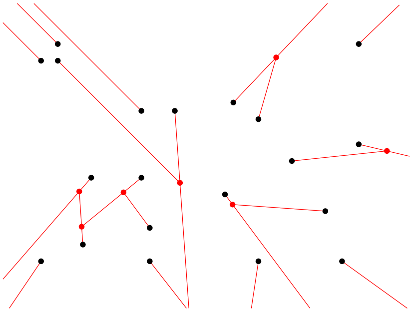

Prior work

- **Trivial brute-force algorithm**
  - Find $O(n)$ crashes in chronological order. Testing each against each takes $O(n^2)$ time for each crash.
  - Using a priority queue results in an $O(n^2 \log n)$ algorithm, instead of $O(n^3)$.

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  - Transformed problem to intersecting 3D faces and considered closest-pair problems.
- **Cheng and Vigneron, 2002**
  - Induced a partitioning of the plane by $1/\sqrt{n}$-cuttings and exploited arrangements on each cutting-cell, resulting in an $O(n\sqrt{n} \log n)$ algorithm.
  - Too complicated for an actual implementation.

Current situation

Summary

- No "close-to linear" algorithm is known.
- No sub-quadratic implementation is known.

Basic idea

Replace the $1/\sqrt{n}$-cutting of Cheng and Vigneron's algorithm by a regular rectangular grid and drop the arrangements.

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Computing motorcycle graphs

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Main question

Consider n motorcycles on a $h \times h$ hash. In the worst case, this leads to $O(n \cdot h)$ crossings of motorcycles with the grid-lines. Do we have a chance to obtain a good performance in practice?

Basic algorithm

Discrete event simulation of the moving motorcycles:

1. Crash event: a motorcycle crashes into a trace.
2. Switch event: a motorcycle leaves one grid cell and enters a neighboring grid cell.

In the course of simulation, the algorithm iteratively extracts the next event from a priority queue and processes it.

Input data and data structures

Our input consists of:

- A set $M := \{m_1, \ldots, m_n\}$ of motorcycles.
  No need to know M a-priori: new motorcycles may emerge, if their start time is in the future.
- A set W of line segments representing walls, where motorcycles may crash against.

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We maintain the following data structures:

- A priority queue Q of pending events.
- For every motorcycle m a binary search trees $C[m]$, where $C[m]$ holds potential future crash events of m.
- A geometric hash H for tracking the motorcycle traces, using a $h \times h$ grid.
- A geometric hash G for the wall-segments, using the same grid geometry as H.

Basic algorithm

```
procedure McGraph(motorcycles M, walls W)
    Q, C, H ← initialize empty
    G ← geometric hash with all w ∈ W

    for all m ∈ M do
        insertMc(m)
                                   ▷ Adds an empty binary tree C[m] to C
                                 ▷ Inserts an initial switch-event of m to Q
    end for

    while not Q.empty() do                          ▷ Process all events e
        e ← Q.pop()
        handle(e)
                       ▷ Switch-event: attach motorcycle to new grid cell,
                                  add next switch-event to Q, maintain C.
                    ▷ Crash-event: clean-up stale future crash-events in C.
    end while
end procedure
```

Runtime complexities

Let k be the maximum number of motorcycles in a hash cell. Processing a switch- resp. crash-event can be done in $O(k \log n)$ time.

There are $O(n)$ crash-events and $O(n \cdot h)$ switch-events and we choose $h \in \Theta(\sqrt{n})$. Hence, the worst case complexity is

$$O(nkh \log n) \subseteq O(n^2 \sqrt{n} \log n).$$

Runtime complexities

Worst case

$\Omega(n)$ motorcycles cross $\Omega(h)$ hash-cells in a narrow strip that is $O(1)$ cells thick. Further, no other motorcycle is allowed to cross this strip before.

. . . what is the expected runtime?

We denote by $S := [0,1]^2$ the unit square covered by a $h \times h$ grid.

Lemma

Let $R = (p, \varphi) \in S \times [0, 2\pi)$ be a uniformly distributed ray, starting at p, with direction angle φ. Further, let C be a cell of a $h \times h$ grid on S. The probability that R intersects C is $\Theta(1/h)$.

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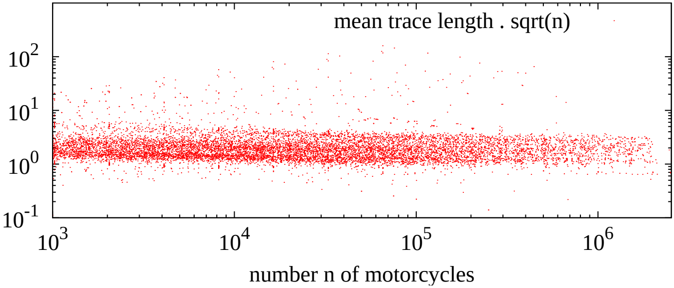
Theorem

Consider n random rays distributed within S. The expected number of rays intersecting a specific cell is in $\Theta(n/h)$.

Expected runtime

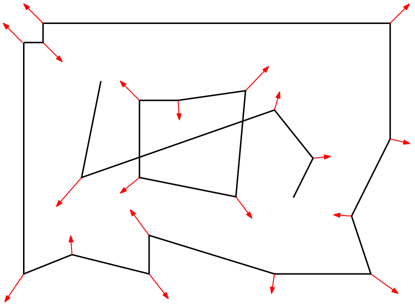


Observation

Consider n random motorcycles within S and let $h \in \Theta(\sqrt{n})$. A motorcycle trace has a mean length proportional to $1/\sqrt{n}$. Hence, it intersects $O(1)$ cells in average. This leads to an $O(n \log n)$ expected runtime.

Experimental setup

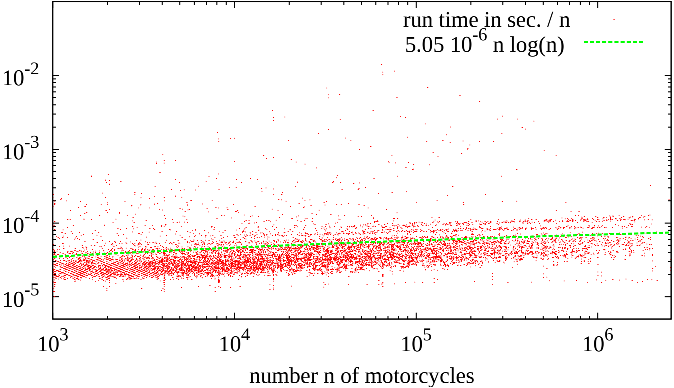
- A data set consist of polygonal chains.
- For every inner vertex of a chain, we define a motorcycle in "straight skeleton" manner. The chains are considered being walls.
- We ran our implementation MOCA on 22 000 thousand data sets, consisting of real-world[1] data and contrived data.



[1] Medical scans, GIS maps, outlines of fonts, CAD/CAM models, etc.

Experimental results

Actual runtime in seconds divided by the number n of motorcylces for each of the 22 000 data sets.



Least-square fit reveals an average run time of $5.05 \cdot 10^{-6} n \log n$ seconds on our computer.

Conclusion

- Easy-to-implement algorithm.
- Surprisingly good performance and competitive in practice.
- Algorithm can be extended easily to more general motorcycle graph problems: motorcycles running out of fuel, curved traces, partial/temporal/conditional invisible traces, etc.

Finish