What makes a Tree a Straight Skeleton?

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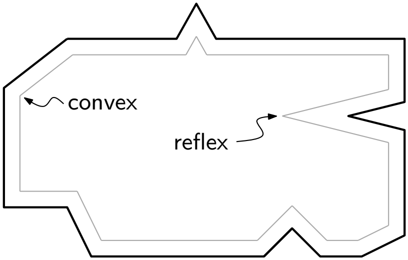
Straight skeletons: an introduction

- Introduced for simple polygons P in [Aichholzer et al., 1995].
- Definition based on **wavefront propagation process**:



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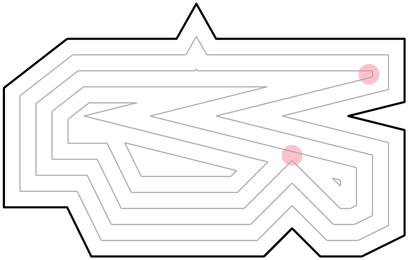

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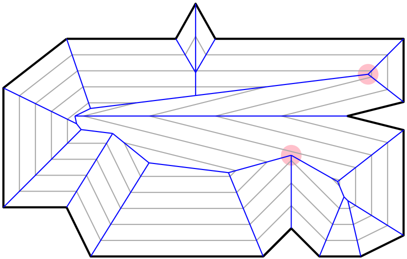
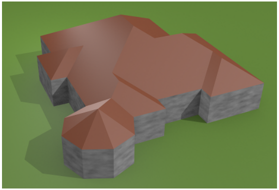
- Definition based on **wavefront propagation process**:
  - edge events,
  - split events.
- Straight skeleton $\mathcal{S}(P)$: set of loci traced out by wavefront vertices.



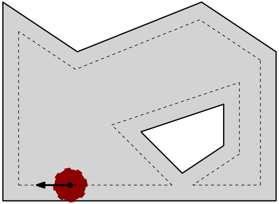
Applications

Straight skeletons have dozens of applications in

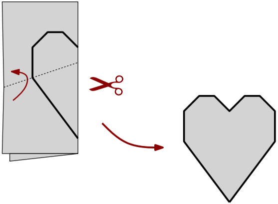
- roof construction, terrain generation
- mitered offset curve computation, tool-path generation
- mathematical origami
- shape reconstruction
- polygon decomposition
- topology-preserving area collapsing in geographic maps
- ...



Roof construction



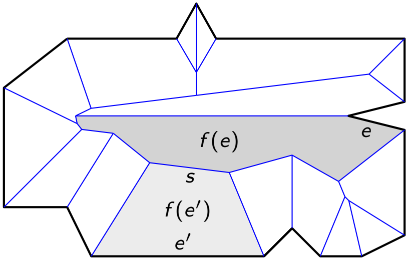
Tool path generation



Fold-and-cut problem

Straight skeletons: basic geometric properties

- P is tessellated into faces.
  - Each face $f(e)$ belongs to an edge e.
- Every straight-skeleton edge s is on the boundary of two faces, $f(e)$ and $f(e')$, and lies on the bisector of e and e'.
- A straight-skeleton vertex v on the boundary of faces $f(e_1), \ldots, f(e_k)$ has equal orthogonal distance to $e_1, \ldots, e_k$.

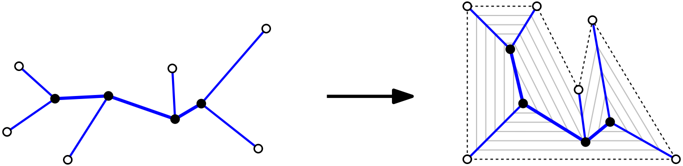


An inverse straight-skeleton problem

We are given:

- a tree (topologically),
- the lengths of the edges,
- at each vertex the circular order of the incident edges.

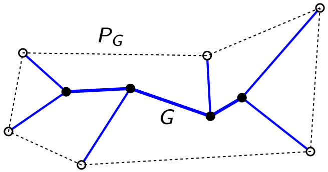
Can we find a polygon P whose straight-skeleton $\mathcal{S}(P)$ matches these requirements?



Problem statement

Notations:

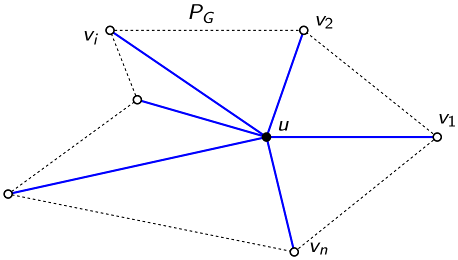
- An abstract geometric graph $\mathcal{G}$ is the set of all geometric graphs with predefined topology, edge lengths and cyclic order of edges at the vertices.
- For a geometric tree $G \in \mathcal{G}$, we denote by P_G the polygon resulting from cyclically connecting the leaves of G.
- We call P suitable for $\mathcal{G}$ if $\mathcal{S}(P) \in \mathcal{G}$.
- We call $\mathcal{G}$ feasible if there is a suitable polygon for $\mathcal{G}$.



- Which $\mathcal{G}$ are feasible?
- If $\mathcal{G}$ is feasible, are the suitable polygons unique?
- How to construct feasible polygons?

Star graphs S_n: introduction

- Let us start with simple trees: star graphs S_n.
  - A vertex u adjacent to n terminal vertices $v_1, \ldots, v_n$.
  - If P_{S_n} is suitable then u has equal orthogonal distance to all polygon edges.



Star graphs S_n: introduction

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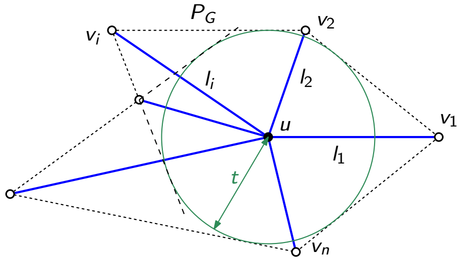
  - A vertex u adjacent to n terminal vertices $v_1, \dots, v_n$.
  - If P_{S_n} is suitable then u has equal orthogonal distance to all polygon edges.
- Hence, there is a tangential circle with some radius t.
  - We denote by l_i the length of uv_i. W.l.o.g. let $l_1 = \max_i l_i$.



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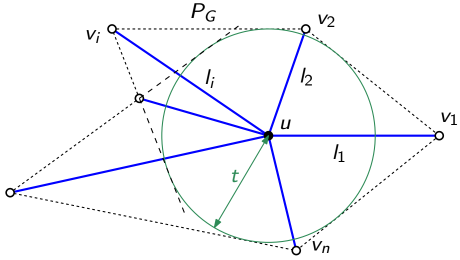

Observation

If P_{S_n} is suitable for S_n then

1. *all straight-skeleton faces are triangles,*
2. *two consecutive vertices cannot be both reflex,*
3. $l_i < l_{i\pm1}$ *for a reflex* v_i*,*
4. *the edges of P_{S_n} have equal orthogonal distance t to u, with $t \leq \min_i l_i$.*

Constructing feasible polygons for S_n

Is there a suitable polygon for S_n for a given convexity/reflexivity assignment A to its vertices?

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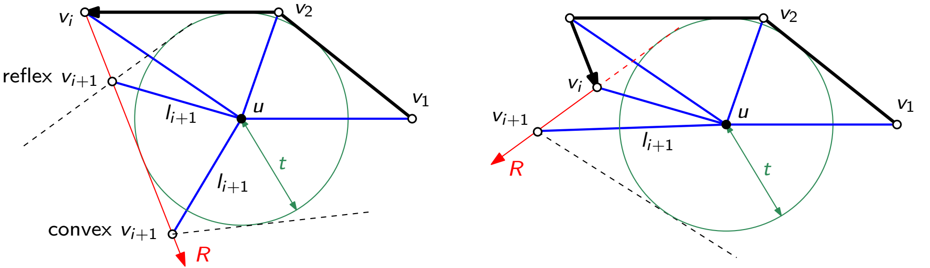
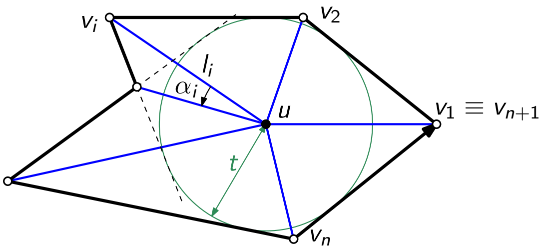
We construct the following polyline $L_{S_n}(t, A)$:

- Place a circle C with radius t and center $u = (0,0)$ and a vertex v_1 at $(l_1, 0)$.
  - v_1 needs to be convex as $l_1 = \max_i l_i$.
- We incrementally construct v_{i+1}, with $1 \leq i \leq n$:
  - Shoot a ray R from v_i s.t.
    (i) the supporting line of R is tangentially to C and
    (ii) C is left to R.
  - Place v_{i+1} on the ray such that v_{i+1} has distance $l_{1+(i \bmod n)}$ to u.



Constructing feasible polygons for S_n

- **Basic idea:** If $L_{S_n}(t, A)$ is closed $(v_{n+1} \equiv v_1)$ and simple then $L_{S_n}(t, A)$ forms a suitable polygon.
- $L_{S_n}(t, A)$ is closed and simple if and only if $\alpha_A(t) := \sum_{i=1}^{n} \alpha_i = 2\pi$.



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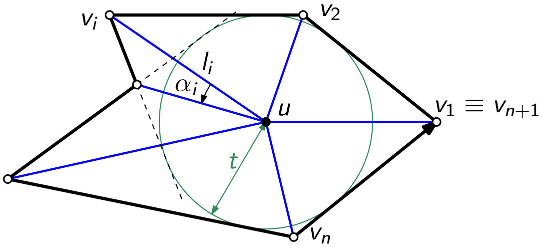

Lemma

$$\alpha_A(t) = 2 \sum_{\substack{i=1 \\ v_i \text{ convex}}}^{n} \arccos \frac{t}{l_i} - 2 \sum_{\substack{i=1 \\ v_i \text{ reflex}}}^{n} \arccos \frac{t}{l_i} \, . \tag{1}$$

Is a star graph S_n feasible?

Lemma

A suitable convex polygon for a star graph S_n exists if and only if $\sum_i \arccos \frac{\min_i l_i}{l_i} \leq \pi$. If a suitable convex polygon exists then it is unique.

Proof idea: Show that a $t \in (0, \min_i l_i]$ exists with $\alpha_A(t) = 2\pi$.

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Proof idea: Show that a $t \in (0, \min_i l_i]$ exists with $\alpha_A(t) = 2\pi$.

Lemma

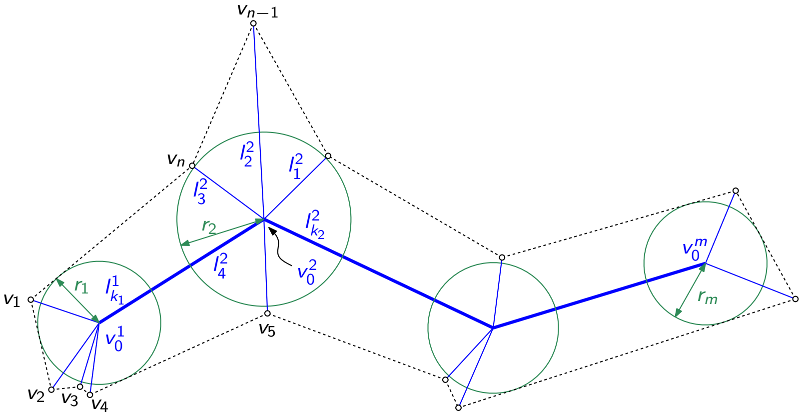
There exist infeasible star graphs S_n. Further, there exist feasible star graphs for which multiple suitable polygons exist.

- S_5 with $l_1 = l_2 = l_3 = l_4 = 1$, and $l_5 = 0.25$ leads to $\alpha_A(t) > 2\pi$ for all valid t and A.
- S_5 with $l_1 = l_3 = 1$, $l_2 = 0.6$, $l_4 = 0.79$, and $l_5 = 0.75$ has two solutions: Assign all vertices convex, except for v_2. Then
  - $t \approx 0.537$ and
  - $t \approx 0.598$

  both result in $\alpha_A(t) = 2\pi$.

Caterpillar graphs: notations

- A caterpillar graph G becomes a path if all leaves are removed.
  - We call this path the backbone (bold).
  - Backbone vertices are denoted by $v_0^1, \ldots, v_0^m$.



Caterpillar graphs: geometric properties

Lemma

The radii $r_2, \ldots, r_m$ for some given caterpillar graph G are determined by r_1 and the edge lengths of G according to the following recursions, for $1 \leq i < m$:

$$r_{i+1} = r_i + l^i_{k_i} \sin \beta_i \tag{2}$$

$$\beta_i = \beta_{i-1} + (1 - {}^{k_i}/_2)\pi + \tag{3}$$

$$\sum_{\substack{j=1 \\ v^i_j \neq v^{i-1}_0}}^{k_i - 1} \begin{cases} \arcsin \frac{r_i}{l^i_j} & \text{if } v^i_j \text{ is convex} \\ \pi - \arcsin \frac{r_i}{l^i_j} & \text{if } v^i_j \text{ is reflex} \end{cases}$$

For $i = 1$ we define that $\beta_0 = 0$ and $v^1_j \neq v^0_0$ being true for all $1 \leq j < k_1$.

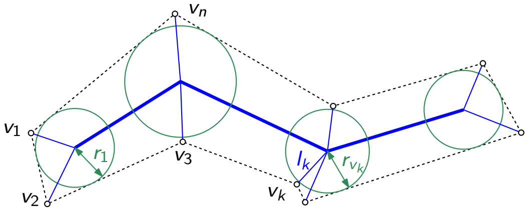
Hence, we can express the sum of the inner angles of P_G as a function of one parameter, r_1.

Caterpillar graphs: feasibility and suitable polygons

Corollary

The sum of the inner angles of P_G with convexity assignment A is a function

$$\alpha_A(r_1) = 2\sum_{j=1}^{n} \begin{cases} \arcsin \frac{r_{v_j}}{l_j} & \text{if } v_j \text{ is convex} \\ \pi - \arcsin \frac{r_{v_j}}{l_j} & \text{if } v_j \text{ is reflex} \end{cases} . \quad (4)$$



Lemma

There is only a finite number of suitable polygons for a caterpillar graph.

Summary

A novel inverse problem: finding polygons for a "given" straight-skeleton graph.

Star graphs S_n:

- We have a tool to decide feasibility.
- We have a construction method for suitable polygons.
- We completely characterized feasibility for S_3 and S_4 (skipped).
- There are infeasible (star) graphs (S_5).
- (Star) graphs (S_5) exist that have multiple suitable polygons.

Caterpillar graphs G:

- We have a tool to decide feasibility.
- We have a construction method for suitable polygons (skipped).
- We know that only finitely many suitable polygons exist.

Bibliography

Aichholzer, O., Alberts, D., Aurenhammer, F., and Gärtner, B. (1995).
Straight Skeletons of Simple Polygons.
In *Proc. 4th Internat. Symp. of LIESMARS*, pages 114–124, Wuhan, P.R. China.

Infeasible star graph S_5

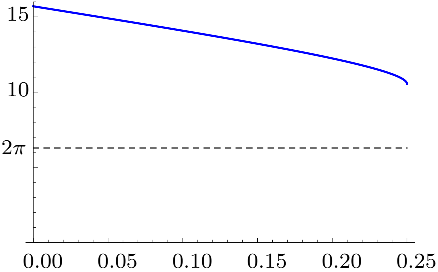


Figure: The sum $\sum_i \alpha_i$ for all $t \in (0, \min_i l_i]$, where $l_1 = \cdots = l_4 = 1$ and $l_5 = 0.25$. Left: v_5 is convex. Right: v_5 is reflex.

Multiple feasible polygons for S_5

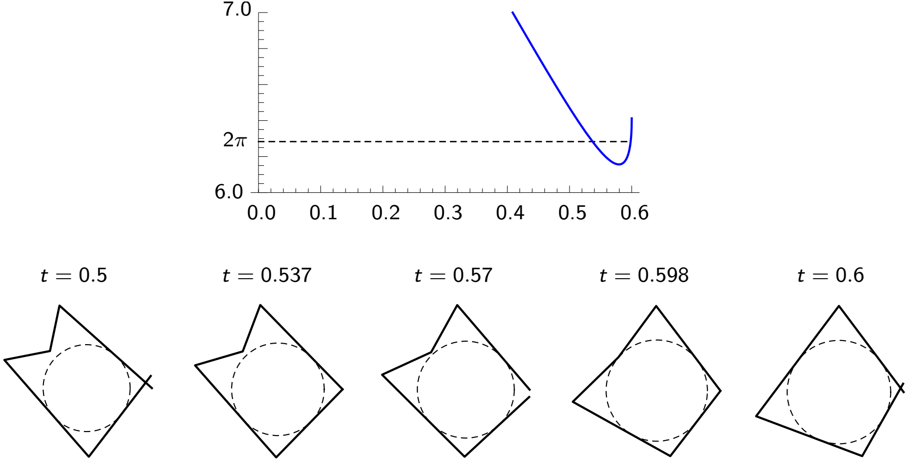


Figure: Edge lengths $l_1 = 0.75$, $l_2 = 1$, $l_3 = 0.6$, $l_4 = 1$, $l_5 = 0.79$. All vertices are convex, except for v_3. Top: $\sum_i \alpha_i$ evaluates to 2π for two different values of t. Bottom: The result of our construction scheme for a sequence of different values of t.