What makes a Tree a Straight Skeleton?*

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Abstract

Let G be a cycle-free connected straight-line graph with predefined edge lengths and fixed order of incident edges around each vertex. We address the problem of deciding whether there exists a simple polygon P such that G is the straight skeleton of P. We show that for given G such a polygon P might not exist, and if it exists it might not be unique. For small star graphs and caterpillars we give necessary and sufficient conditions for constructing P.

1 Introduction

The straight skeleton $\mathcal{S}(P)$ of a simple polygon P is a skeleton structure like Voronoi diagrams, but consists of straight-line segments only. Its definition is based on a so-called *wavefront propagation* process that corresponds to mitered offset curves. Each edge e of P emits a wavefront that moves with unit speed to the interior of P. Initially, the wavefront of P consists of parallel copies of edges of P. However, during the wavefront propagation, topological changes occur: An *edge event* happens if a wavefront edge shrinks to zero length. A *split event* happens if a reflex wavefront vertex meets a wavefront edge and splits the wavefront into pieces, see Figure 1 (right). The *straight skeleton* $\mathcal{S}(P)$ is defined as the set of loci that are traced out by the wavefront vertices and it partitions P into polygonal faces. Each face $f(e)$ belongs to a unique edge e of P. Each straight skeleton edge belongs to two faces, say $f(e_1)$ and $f(e_2)$, and lies on the bisector of e_1 and e_2. Straight skeletons have many applications, like automated roof construction, computation of mitered offset curves, and solving fold-and-cut problems. See [4] and Chapter 5.2 in [3] for further information and detailed definitions.

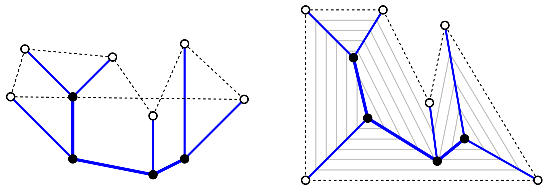

Figure 1: Example of a feasible cycle-free connected abstract geometric graph G (leaves of G are shown as white dots). Left: Arbitrary embedding $E(G)$ and (non-simple) polygon $P_{E(G)}$ (dotted). Right: Suitable polygon $P_{E'(G)}$ for a different embedding $E'(G)$, which is equal to $\mathcal{S}(P_{E'(G)})$. A set of wavefronts of $P_{E'(G)}$ at different points in time are depicted in gray.

Although straight skeletons were introduced to computational geometry in 1995 by Aichholzer et al. [1], their roots actually go back to the 19th century. In textbooks about the construction of roofs (see e.g. [6], pages 86–122) using the angle bisectors (of the polygon defined by the ground walls) was suggested to design roofs where rainwater can run off in a controlled way. This construction is called *Dachausmittlung* and became rather popular. See [5] for related and partially more involved methods to obtain roofs from the ground plan of a house.

Maybe not surprisingly, none of this early works mentions the ambiguity of the non-algorithmic definition of the construction. Using solely bisector graphs does not necessarily lead to a unique roof construction, and actually not even guarantees a plane partition of the interior of the defining boundary [1].

An interesting inverse problem was motivated to us by Lior Pachter and investigations started in [2]: Which graphs are the straight skeleton of some polygon? To give a more formal problem definition we denote with *abstract geometric graphs* the set of combinatorial graphs, where the length of each edge and the cyclic order of incident edges around every vertex is predefined (and cannot be altered). Let $\mathcal{G}$ be the set of cycle-free connected abstract geometric graphs. Denote with $E(G)$ an embedding of $G \in \mathcal{G}$ in the plane, that is, the vertices of G are points in $\mathbb{R}^2$ and the edges of G are straight-line segments of the predefined length, connecting the corresponding points

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and respecting the predefined cyclic order of incident edges around each vertex. Further, denote with $P_{E(G)}$ the polygon resulting from connecting the leaves of G (with straight-line segments) in cyclic order for the embedding $E(G)$. We call a simple polygon $P_{E(G)}$ *suitable* if its straight skeleton $\mathcal{S}(P_{E(G)}) = E(G)$, for the embedding $E(G)$. If there exists a suitable polygon for a graph $G \in \mathcal{G}$, we call G *feasible*, see Figure 1.

The obvious questions which arise from these definitions are: Which graphs of $\mathcal{G}$ are feasible? Are the suitable polygons for feasible graphs unique? How can one construct a suitable polygon for a feasible graph?

2 Star graphs

All polygon edges whose straight-skeleton faces contain a common vertex u (of the straight skeleton) have equal orthogonal distance t to u, because their wavefront edges reach u at the same time t. That is, the supporting lines of those polygon edges are tangential to the circle with center u and radius t. Thus, in this section we consider a subset of $\mathcal{G}$, the so called star graphs. A *star graph* $S_n \in \mathcal{G}$, for $n \geq 3$ has $(n+1)$ vertices, one vertex u with degree n and n leaves $v_1, \ldots, v_n$ ordered counter clockwise around u. The length of each edge uv_i, with $1 \leq i \leq n$, is denoted by l_i. W.l.o.g. let $l_1 = \max_i l_i$. Observe that the polygon $P_{E(S_n)}$ is star shaped and $v_i v_{i+1}$ (with $v_{n+k} := v_{1+(k-1) \bmod n}$) are its edges.

Observation 1 *If $S_n \in \mathcal{G}$ is a feasible star graph and $P_{E(S_n)}$ is a suitable polygon of S_n, then (1) all straight-skeleton faces are triangles, (2) two consecutive vertices v_i, v_{i+1} can not be both reflex, (3) $l_i < l_{i\pm1}$ for each reflex vertex v_i of $P_{E(S_n)}$, and (4) all edges of $P_{E(S_n)}$ have equal orthogonal distance t to u, with $t \in (0, \min_i l_i]$.*

As a given $S_n \in \mathcal{G}$ is possibly not feasible and a suitable polygon may not be known or does not exist, we define a polyline $L_{S_n}(t, A)$: The vertices $v_1, \ldots, v_{n+1}$ of $L_{S_n}(t, A)$ are the leaves, $v_1, \ldots, v_n$, of S_n, in the same order as for S_n, and one additional vertex v_{n+1} succeeding v_n. The vertices $v_1, \ldots, v_n, v_{n+1}$ have the corresponding distances (predefined in S_n) $l_1, \ldots, l_n, l_1$ to u. A is an assignment for each vertex whether it should be convex or reflex, as seen from u. As $l_1 = \max_i l_i$, v_1 and v_{n+1} are always convex (fact (3) in Observation 1). For the remaining vertices any convex/reflex assignment, which respects the facts (2) and (3) in Observation 1, can be considered. The edges of $L_{S_n}(t, A)$ have equal orthogonal distance t to u. Of course, not all possible combinations of t and an arbitrary embedding $E(S_n)$ allow such a polyline. But it is possible to construct $L_{S_n}(t, A)$ and $E(S_n)$ simultaneously for a fixed $t \in (0, \min_i l_i]$.

For a fixed assignment A and a fixed $t \in (0, \min_i l_i]$ we construct $L_{S_n}(t, A)$ (and $E(S_n)$) in the following way. Consider the circle C with center u and radius t. Start with v_1 at polar coordinate $(l_1, 0)$, with u as origin. For each v_i, $i = 2 \ldots (n+1)$, consider a tangent g_{i-1} to C (such that the vertices will be placed counter clockwise around the circle) through v_{i-1}. If v_{i-1} is convex, then there exist two points with distance l_i (l_1 for v_{n+1}) on g_{i-1}. If v_i is assigned to be reflex, then v_i is placed on the point closer to v_{i-1}, and if v_i is assigned to be convex, then v_i is placed on the other point. If v_{i-1} is reflex, then there exists only one applicable point for placing v_i on g_{i-1}. See Figure 2.

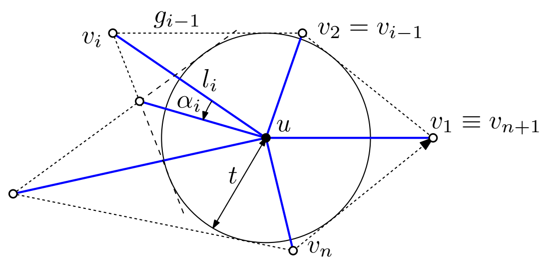

Figure 2: Construction of $L_{S_n}(t, A)$ (and $E(S_n)$) for a given S_n and a fixed distance t and assignment A.

The $L_{S_n}(t, A)$ constructed this way is unique (for fixed t and A), and may be not simple (e.g. when circling C many times), simple but not closed ($v_{n+1} \not\equiv v_1$), or simple and closed ($v_{n+1} \equiv v_1$). In the latter case, the construction reveals a witness pair (t, A) for the existence of some $E(S_n)$, a suitable polygon $P_{E(S_n)}$, and thus the feasibility of S_n.

It is easy to see, that for each suitable polygon $P_{E(S_n)}$, there exists a polyline $L_{S_n}(t, A)$ (just duplicate the vertex v_1). Hence, deciding feasibility of S_n is equivalent to finding an assignment A and a $t \in (0, \min_i l_i]$ such that $L_{S_n}(t, A)$ is closed and simple. For a polyline $L_{S_n}(t, A)$ and a corresponding embedding $E(S_n)$, we denote with α_i, $i = 1 \ldots n$, the counter clockwise angle at u, spanned by uv_i and uv_{i+1}. (Note that for a suitable polygon $P_{E(S_n)}$ α_i can be defined the same way, with $v_{n+1} \equiv v_1$.) It is easy to see that the sum of all α_i is 2π if and only if $L_{S_n}(t, A)$ is closed and simple.

Lemma 1 *Let $S_n \in \mathcal{G}$, distance $t \in (0, \min_i l_i]$ and assignment A be fixed, and let $L_{S_n}(t, A)$ be the resulting polyline. Then $\alpha_A(t) := \sum_{i=1}^{n} \alpha_i$ can be expressed as*

$$\alpha_A(t) = 2 \sum_{\substack{i=1 \\ v_i \text{ convex}}}^{n} \arccos \frac{t}{l_i} - 2 \sum_{\substack{i=1 \\ v_i \text{ reflex}}}^{n} \arccos \frac{t}{l_i} \ . \quad (1)$$

Proof. Omitted in this version. □

For the following result we use the first derivative of α_A:

$$\alpha'_A(t) = 2 \sum_{\substack{i=1 \\ v_i \text{ reflex}}}^{n} \frac{1}{\sqrt{l_i^2 - t^2}} - 2 \sum_{\substack{i=1 \\ v_i \text{ convex}}}^{n} \frac{1}{\sqrt{l_i^2 - t^2}}. \quad (2)$$

Lemma 2 *A suitable convex polygon for a star graph S_n exists if and only if $\sum_i \arccos \frac{\min_i l_i}{l_i} \leq \pi$. If a suitable convex polygon exists then it is unique.*

Proof. As all vertices are assumed to be convex, we obtain $\alpha_A(0) = n\pi > 2\pi$. Furthermore, we observe that $\alpha_A(t)$ is monotonically decreasing since $\alpha'_A(t) < 0$ for all $t \in (0, \min_i l_i]$. Hence, there is a $t \in (0, \min_i l_i]$ with $\alpha(t) = 2\pi$ if and only if $\alpha_A(\min_i l_i) \leq 2\pi$ which is $\sum_i \arccos \frac{\min_i l_i}{l_i} \leq \pi$. If this is the case the solution is unique as $\alpha(t)$ is monotonic. ☐

For $n = 3$, $\alpha_A(0) = 3\pi$ and $\alpha_A(\min_i l_i) < 2\pi$, and thus we immediately get the following corollary.

Corollary 3 *For every S_3 there exists a unique suitable convex polygon.*

Considering star graphs with $n = 5$, we show in the following lemma that they are not always feasible, and that suitable polygons (if they exist) are not always unique.

Lemma 4 *There exist infeasible star graphs, $S_n \in \mathcal{G}$. Further, there exist feasible star graphs for which multiple suitable polygons exist.*

Proof. To prove the first claim consider a star graph with $n = 5$, $l_1 = l_2 = l_3 = l_4 = 1$, and $l_5 = 0.25$. There exist only two possible assignments: either all vertices convex or all but v_5 convex. It is easy to check that for both assignments $\sum_i \alpha_i > 2\pi$, for every $t \in (0, \min_i l_i]$. To prove the second claim consider a star graph with $n = 5$, $l_1 = l_3 = 1$, $l_2 = 0.6$, $l_4 = 0.79$, and $l_5 = 0.75$. Assign all vertices convex, except for v_2. Then $\sum_i \alpha_i$ evaluates to 2π for $t \approx 0.537$ and $t \approx 0.598$. Hence, there exist (at least) two different suitable polygons for this star graph. ☐

In the following we discuss sufficient and necessary conditions for the feasibility of a star graph S_4. By Lemma 2 we know in which cases suitable convex polygons exist. The remaining cases are solved by the following lemma.

Lemma 5 *Consider an S_4 for which no suitable convex polygon exists. A suitable non-convex polygon exists if and only if $\frac{1}{\min_i l_i} < \sum_{j=1, l_j \neq \min_i l_i} \frac{1}{l_j}$.*

Proof. First of all, if a polyline has two or more reflex vertices assigned then $\alpha_A(t) < 2\pi$, as each positive summand in Equation (1) is bound by $\pi/2$. Hence, we only need to consider polylines with exactly one reflex vertex, which implies $\alpha_A(0) = 2\pi$.

For simplicity, we may reorder v_i and l_i such that $l_4 = \min_i l_i$. We show that for suitable non-convex polygons v_4 needs to be reflex. Assume to the contrary that some v_k, with $1 \leq k \leq 3$, is reflex. In this case we obtain that $\alpha'_A(t) < 0$ as $1/\sqrt{l_4^2 - t^2}$ dominates $1/\sqrt{l_k^2 - t^2}$ for all $t \in [0, l_4)$. But since $\alpha_A(0) = 2\pi$ we see that $\alpha_A(t) < 2\pi$ for all $t \in (0, \min_i l_i]$.

Observe that the assumption in the lemma, that no suitable convex polygon exists, is equivalent to $\alpha_A(l_4) > 2\pi$. Recall that $\alpha_A(0) = 2\pi$. Hence, if $\alpha'_A(0) < 0$ then there exists a $t \in (0, l_4)$ such that $\alpha_A(t) = 2\pi$, as α_A is continuously differentiable.

Finally, we show that if $\alpha'_A(0) \geq 0$ then $\alpha'_A(t) > 0$ for all $t \in (0, l_4)$. Hence, there is no $t \in (0, l_4]$ such that $\alpha_A(t) = 2\pi$. From Equation (2) we get that $\alpha'_A(t) > 0$ is equivalent to

$$\frac{1}{\sqrt{l_4^2 - t^2}} > \sum_{i=1}^{3} \frac{1}{\sqrt{l_i^2 - t^2}} \quad \Leftrightarrow \quad 1 > \sum_{i=1}^{3} \sqrt{1 - \frac{l_i^2 - l_4^2}{l_i^2 - t^2}}$$

The right side of this equivalence is true since

$$1 \geq \sum_{i=1}^{3} \sqrt{1 - \frac{l_i^2 - l_4^2}{l_i^2}} > \sum_{i=1}^{3} \sqrt{1 - \frac{l_i^2 - l_4^2}{l_i^2 - t^2}}, \qquad (3)$$

where the first inequality is given by $\alpha'_A(0) \geq 0$ and the second inequality holds for all $t \in (0, l_4)$.

To conclude, we have shown that if no suitable convex polygon exists for some S_4, then a suitable non-convex polygon exists for this S_4 if and only if $\alpha'(0) < 0$, which is equivalent to $\frac{1}{\min_i l_i} < \sum_{j=1, l_j \neq \min_i l_i} \frac{1}{l_j}$, as claimed in the lemma. ☐

3 Caterpillar graphs

The techniques developed in the previous section can be generalized to so-called caterpillar graphs. A *caterpillar graph* $G \in \mathcal{G}$ is a graph that becomes a path if all its leaves (and their incident edges) are removed. We call this path the *backbone* of G. Figure 1 shows a caterpillar graph whose backbone comprises three backbone edges.

In general, a caterpillar graph has m backbone vertices, consecutively denoted by $v_0^1, \ldots, v_0^m$. We denote the adjacent vertices of a backbone vertex v_0^i, with k_i incident edges, by $v_1^i, \ldots, v_{k_i}^i$, such that $v_{k_i}^i = v_0^{i+1}$ for $1 \leq i < m$. Furthermore, we denote by l_j^i the length of the edge $v_0^i v_j^i$, see Figure 3. Let us consider a polygon P whose straight skeleton $\mathcal{S}(P)$ forms a caterpillar graph.

Observation 2 *All edges of P whose straight-skeleton faces contain the same backbone vertex v_0^i have identical orthogonal distance to v_0^i.*

We denote this orthogonal distance by r_i. Hence, the supporting lines of the corresponding polygon edges are tangents to the circle of radius r_i centered at v_0^i, see Figure 3.

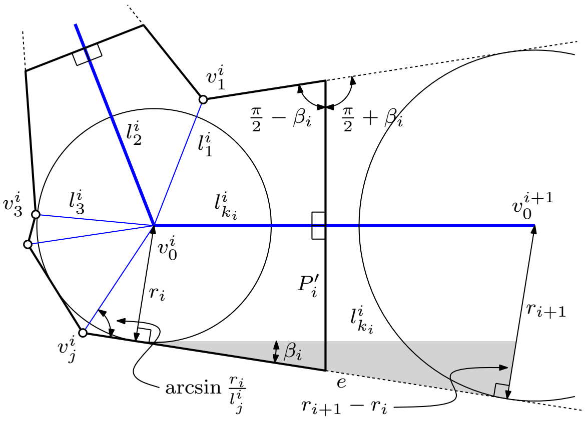

Figure 3: A section of a polygon P for which $\mathcal{S}(P)$ is a caterpillar graph.

Lemma 6 *The radii $r_2, \ldots, r_m$ of a suitable polygon $P_{E(G)}$ for some given caterpillar graph G are determined by r_1 and the predefined edge lengths of G according to the following recursions, for $1 \le i < m$:*

$$\begin{aligned} r_{i+1} &= r_i + l_{k_i}^i \sin \beta_i \\ \beta_i &= \beta_{i-1} + (1 - {}^{k_i}\!/_2)\pi + \\ &\sum_{\substack{j=1 \\ v_j^i \neq v_0^{i-1}}}^{k_i - 1} \begin{cases} \arcsin \frac{r_i}{l_j^i} & v_j^i \text{ is convex} \\ \pi - \arcsin \frac{r_i}{l_j^i} & v_j^i \text{ is reflex} \end{cases} \end{aligned}$$

For $i = 1$ we define that $\beta_0 = 0$ and $v_j^1 \neq v_0^0$ being true for all $1 \le j < k_1$.

Proof. Denote with e one of the two edges of $P_{E(G)}$ whose faces of $\mathcal{S}(P_{E(G)})$ contain the edge $v_0^i v_0^{i+1}$. The supporting line of e is tangential to the circles at v_0^i and v_0^{i+1}. Considering the shaded right-angled triangle in Figure 3, we obtain $r_{i+1} - r_i = l_{k_i}^i \cdot \sin \beta_i$.

Consider the polygon P_i' (bold in Figure 3) which comprises the edges of $P_{E(G)}$ whose faces of $\mathcal{S}(P_{E(G)})$ contain v_0^i, trimmed by two additional edges orthogonal to $v_0^{i-1} v_0^i$ and $v_0^i v_0^{i+1}$, respectively. P_i' comprises k_i+2 vertices (k_1+1 for P_1') and hence, the sum of inner angles equals $k_i \pi$ ((k_1−1)π for P_1'). On the other hand, we can express this sum as follows (also for P_1'), which implies the second recursion:

$$\begin{aligned} k_i \pi = 2\pi + 2\beta_{i-1} - 2\beta_i + \\ 2 \sum_{\substack{j=1 \\ v_j^i \neq v_0^{i-1}}}^{k_i - 1} \begin{cases} \arcsin \frac{r_i}{l_j^i} & v_j^i \text{ is convex} \\ \pi - \arcsin \frac{r_i}{l_j^i} & v_j^i \text{ is reflex} \end{cases} \end{aligned}$$

□

Corollary 7 *The sum of the inner angles of $P_{E(G)}$ with convexity assignment A is a function*

$$\alpha_A(r_1) = 2 \sum_{j=1}^{n} \begin{cases} \arcsin \frac{r_{v_j}}{l_j} & v_j \text{ is convex} \\ \pi - \arcsin \frac{r_{v_j}}{l_j} & v_j \text{ is reflex} \end{cases}, \quad (4)$$

where r_{v_j} denotes the radius of the circle at the backbone vertex that is adjacent to v_j and l_j denotes the length of the incident edge of G.

The previous corollary provides us with a tool in order to find suitable polygons $P_{E(G)}$ for caterpillar graphs G. We know that for any suitable polygon $P_{E(G)}$ the identity $\alpha_A(r_1) = (n-2)\pi$ must hold. Hence, we can determine all suitable polygons $P_{E(G)}$ as follows: for all 2^n possible assignments A we determine all r_1 such that $\alpha_A(r_1) = (n-2)\pi$.

For any such pair (A, r_1) we construct a polyline $v_1, \ldots, v_n, v_{n+1}$ by a similar method as outlined for star graphs: shooting rays tangential to circles centered at the backbone vertices v_0^i. In order to switch over from v_0^i to v_0^{i+1}, we consider the previously constructed ray, which needs to be tangential to the two circles centered at both, v_0^i and v_0^{i+1}, respectively. As the length of the edge $v_0^i v_0^{i+1}$ is given, the center v_0^{i+1} of the next circle is uniquely determined, cf. Figure 3. If there is any non-backbone edge with length $l_j^i < r_i$ then there is no suitable polygon for that particular pair (A, r_1). For each candidate polyline we check whether it is closed, simple and forms a suitable polygon. Note that all suitable polygons can be constructed by the above method.

Lemma 8 *There is at most a finite number of suitable polygons $P_{E(G)}$ for a caterpillar graph G.*

Proof. As α_A is analytic, there are no accumulation points in the set $\{r_1 : \alpha_A(r_1) = (n-2)\pi\}$. Otherwise, α_A would be identical to $(n-2)\pi$. In other words, there is only a finite number of possible pairs (A, r_1) that correspond to a suitable polygon. □

References

[1] O. Aichholzer, D. Alberts, F. Aurenhammer, and B. Gärtner. *A novel type of skeleton for polygons.* Journal of Universal Computer Science, 1(12):752–761, 1995.

[2] H. Cheng, S.L. Devadoss, B. Li, A. Risteski. *Skeletal rigidity of phylogenetic trees.* preprint 2011.

[3] S.L. Devadoss, J. O'Rourke. *Discrete and Computational Geometry.* Princeton University Press, Princeton and Oxford, 2011.

[4] S. Huber. Computing Straight Skeletons and Motorcycle Graphs: Theory and Practice. PhD thesis, University of Salzburg, Austria, 2011.

[5] E. Müller. *Lehrbuch der darstellenden Geometrie für technische Hochschulen.* Band 2, Verlag B.G.Teubner, Leipzig & Berlin, 1916.

[6] G.A.V. Peschka. *Kotirte Ebenen und deren Anwendung.* Verlag Buschak & Irrgang, Brünn, 1877.