The Topology of Skeletons and Offsets

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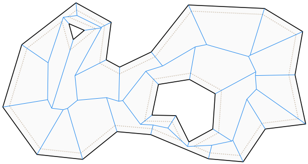
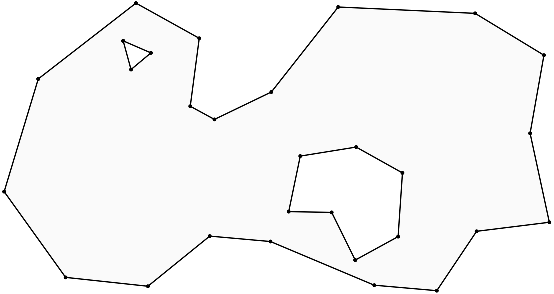


Input:

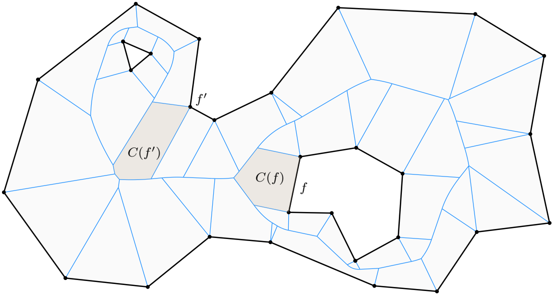
- Polygon with holes, P.
  - $\operatorname{bd} P$ ist set of disjoint closed polygonal curves.
  - $\operatorname{bd} P$ has faces (vertices & edges), a complex.



Skeletons and (their) offsets:

- Voronoi diagram
- Straight skeleton

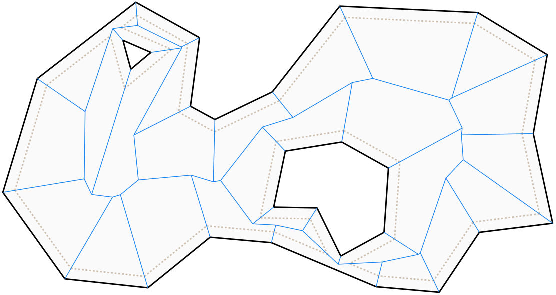
- Nearest-neighbor cell decomposition of P. Each face of bd P has a cell.
- Voronoi cell $C_V(f)$ contains the points closer to f than any other face.
- Voronoi diagram $V(P)$ formed by boundaries of Voronoi cells.
  - Apex-splitting as in Held and Huber (2009)[1].



[1] M. Held and S. Huber. "Topology-Oriented Incremental Computation of Voronoi Diagrams of Circular Arcs and Straight-Line Segments." In: *Comp. Aided Design* 41.5 (May 2009), pp. 327–338. DOI: 10.1016/j.cad.2008.08.004.

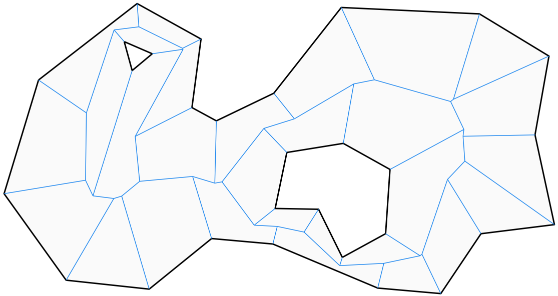
Introduction: Straight skeletons

- bd P emanates wavefront towards interior.
- Wavefront edges collapse or get split.
- The traces of the wavefront vertices form straight skeleton $S(P)$.
- The area swept out by the wavefront of f is called its straight-skeleton cell $C_S(f)$.



We have:

- Some “line structure” on P inducing some cell-decomposition of P.
- Those “line structures” on P capture geometrical and topological features of P.
  - If P is simple then the “line structure” is a tree.
  - If we punch a hole into P we get a new (generator) cycle (in a group of cycles).



What makes a “line structures” a *skeleton* is *homotopy equivalence* to P.

- A skeleton encodes *the* topology of P.

Theorem

The homotopy equivalences $P \simeq S(P) \simeq V(P)$ hold.

Lemma

$S(P)$ is a deformation retract of P, and hence $P \simeq S(P)$.

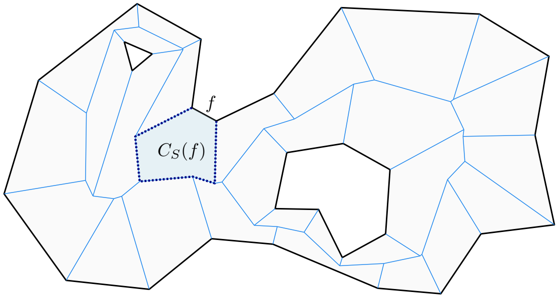
Lemma

$V(P)$ is a deformation retract of P, and hence $P \simeq V(P)$.

Proof.

- $C_S(f)$ is a topological disk. $S(P,f) := S(P) \cap C_S(f)$.
- $C_S(f)$ can be deformation retracted to $S(P,f)$ s.t. it stays constant on $S(P,f)$.
- Plug together, get a def.ret. of $P = \bigcup_f C(f)$ to $\bigcup_f S(P,f) = S(P)$.

□



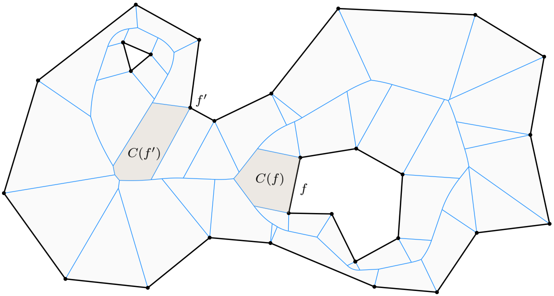
Works for positively-weighted straight skeletons, but not if negative weights are allowed.[2]

[2] T. Biedl, M. Held, S. Huber, D. Kaaser, and P. Palfrader. "Weighted Straight Skeletons In the Plane." In: *Comp. Geom. Theory & Appl.* 48.2 (Feb. 2015), pp. 120–133. DOI: 10.1016/j.comgeo.2014.08.006.

Same proof technique as for $S(P)$ applies.

Technical detail:

- Voronoi edges at reflex vertices f' are considered topologically disjoint.
- As in Held and Huber (2009). Algorithmically handy, topologically natural.
- Hence, $V(P, f')$ is a topological line, not a circle.



Alternative:

- Use Lieutier (2004)[3]: $M(P) \simeq P$, for medial axis $M(P)$.
- Argue that $M(P) \simeq V(P)$, see details above.

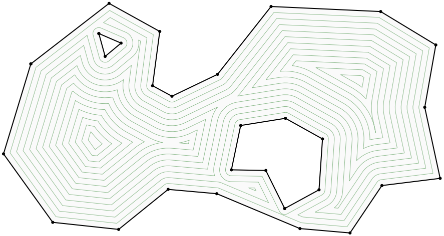
[3] A. Lieutier. "Any Open Bounded Subset of $\mathbb{R}^n$ Has The Same Homotopy Type Than Its Medial Axis." In: *Comp. Aided Design* 36.11 (Sept. 2004), pp. 1029–1046. DOI: 10.1016/j.cad.2004.01.011.

Corollaries

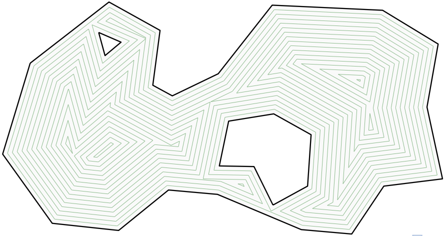
- $P, S(P), V(P)$ are homologous.
- Isomorphic fundamental group, i.e. "holes $\simeq$ cycles".
- $P, S(P), V(P)$ have same Euler characteristics: $2 - 2h$.
  - h the number of holes of P.

Two types of offset curves

- Minkowski-based offset curves

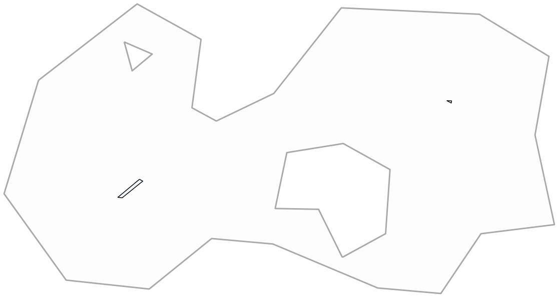


- Mitered offset curves



Homological evolution of offset shapes

We denote by $Q_S(r)$ resp. $Q_V(r)$ the mitered- resp. Minkowski-inset polygon by offset r.



Persistent homology investigates evolution homology groups:

- Offset filtration of polygons: $Q_*(r_0) \subset \cdots \subset Q_*(r_k)$ with $r_0 \geq \ldots \geq r_k = 0$.
- Components and holes in filtration are born and killed (merged).

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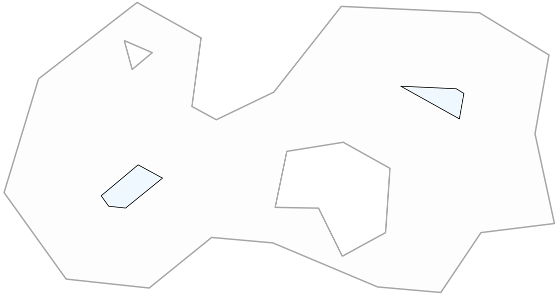

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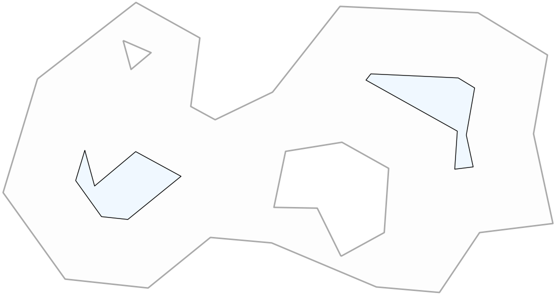

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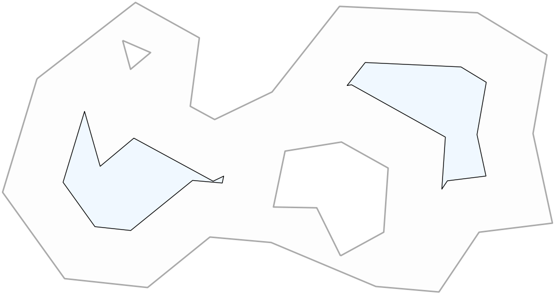

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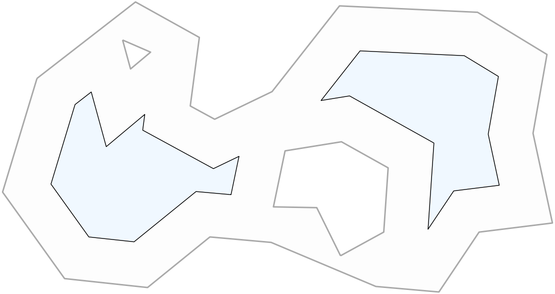

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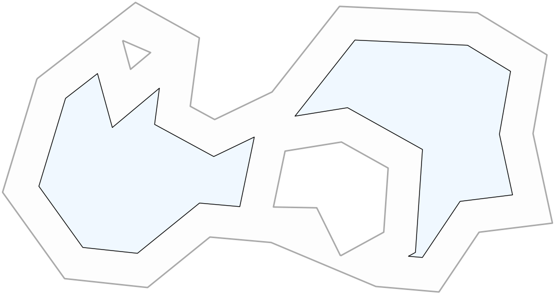

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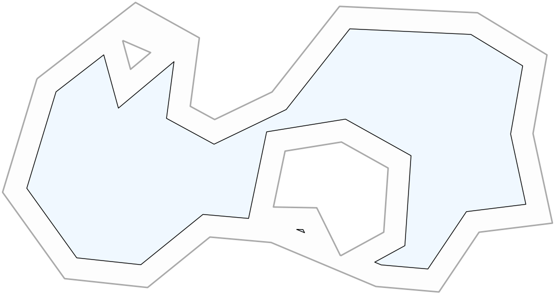

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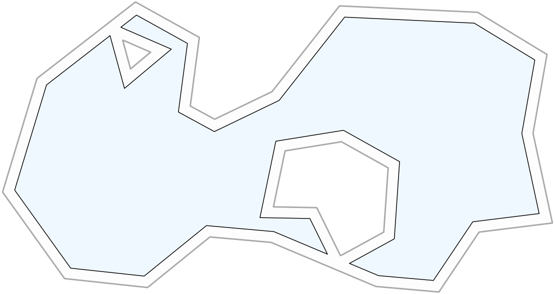

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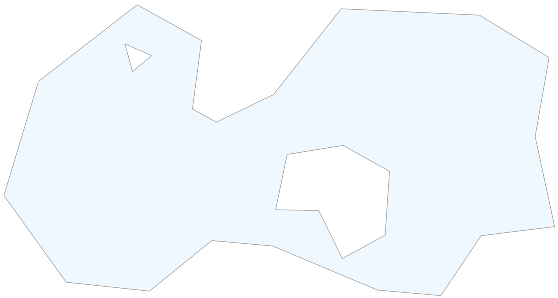

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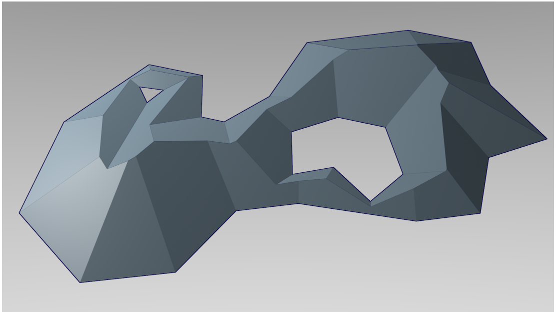
Raindrop property

Consider roof model $R_S(P) = \bigcup_{r \geq 0} \operatorname{bd} Q_S(r) \times \{r\} \subset \mathbb{R}^3$; same for $R_V(P)$.



Lemma

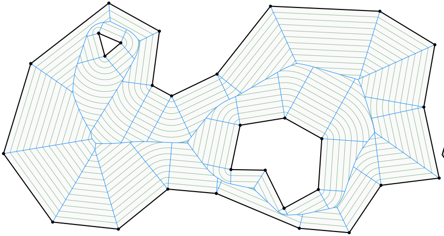
$R_S(P)$ and $R_V(P)$ do not possess local minima, except on $\operatorname{bd} P \times \{0\}$.

Corollary

1-dimensional homology classes never die in the offset filtration.

Duality of skeleton and offset curves:

- Easy computation of offset curves from skeleton.
- Skeleton can be obtained from evolution of offset curves.



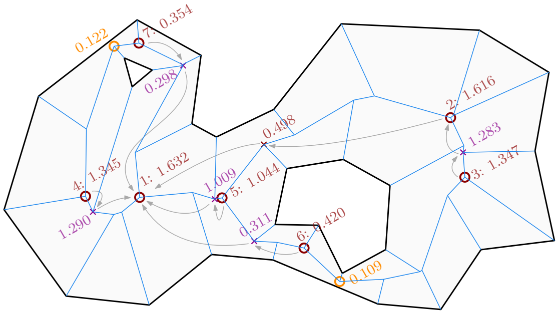
Lemma

$S(Q_S(r)) = S(P) \cap Q_S(r)$, hence $Q_S(r) \simeq S(P) \cap Q_S(r)$. Same for Voronoi diagram.

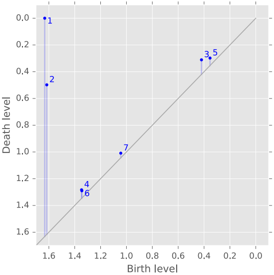
Insert vertices of skeleton in reversed offset order.

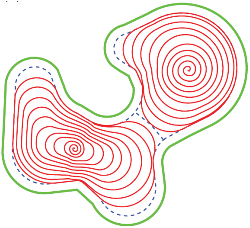
Two cases when inserting vertex v:

1. No neighbor was inserted yet. New component is born.
2. Neighbors $u_1, \ldots, u_d$ were already inserted.
   - Every connected component with c vertices in neighbor set: $c-1$ cycles are closed.
   - All involved components are merged to the oldest one, including v.



After sorting, takes $O(n\alpha(n))$ for computing birth and death of homology classes.





https://www.cosy.sbg.ac.at/~held

Machine learning

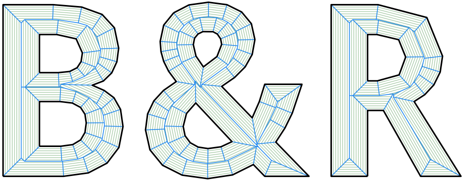
- Persistence diagrams:
  - Multi-set of points.
  - Each point depicts a homology class.
- Kernel on persistence diagrams.[4]
- Kernel-based SVM, k-means, PCA.
- $\rightarrow$ Classification of polygons

Maximum inscribed circle

- Quantification of significance of peaks.
- $\rightarrow$ Polygon decomposition algorithms
- $\rightarrow$ High-speed spiral tool-path planing

[4] R. Kwitt, U. Bauer, S. Huber, M. Niethammer, and W. Lin. “Statistical Topological Data Analysis – A Kernel Perspective.” In: *Proc. 29th Conf. Neural Inf. Proc. Sys. (NIPS '15)*. Montreal, Canada: Curran Associates, Dec. 2015

Thank you



References

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We define the straight-skeleton roof model

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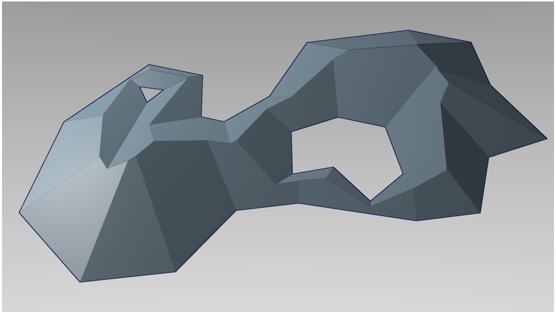
and accordingly for $R_V(P)$.



Could triangulate the onion layers between offset curves and apply boundary algorithm for persistence: $O(m^3)$ runtime, with $m \in O(n^2)$ being size of triangulation.