Planar Matchings for Weighted Straight Skeletons

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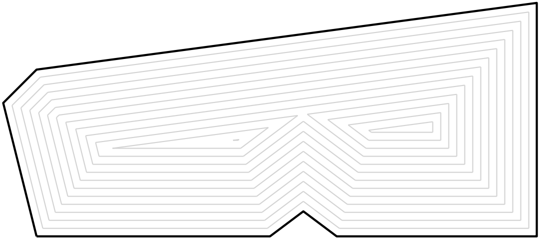
Straight skeletons: A brief introduction



- Introduced by [Aichholzer et al., 1995].

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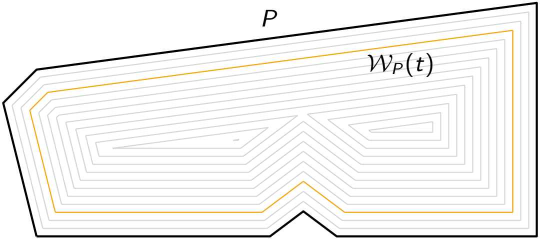

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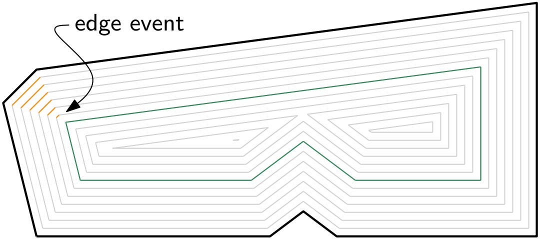

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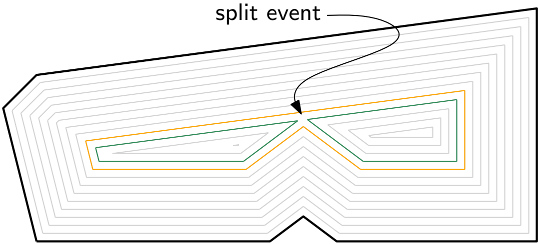

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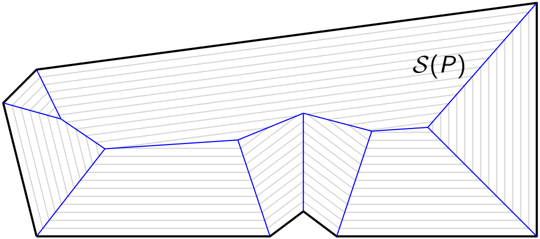
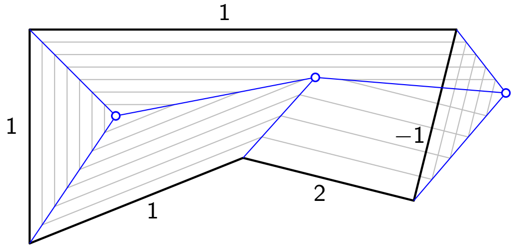


- Introduced by [Aichholzer et al., 1995].
- Generalized to polygons with holes and PSLGs.

Weighted straight skeletons

- First studied in [Eppstein and Erickson, 1999].
- To every edge e of P a weight $\sigma(e) \neq 0$ is assigned, its speed.



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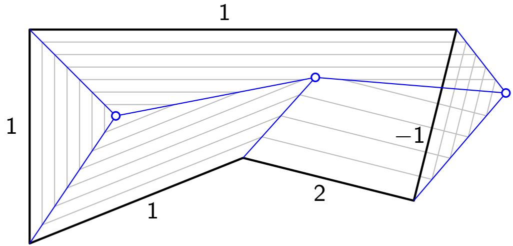


Weighted straight skeletons are "quite established":

- Algorithms were published.
- Implementations are available.
- Used in theory & applications.

Still no rigorous definition is known so far!

Prior work

It was silently assumed that weights would not change much.

- Recently we showed that this is not quite true.

Property	Simple polygon $\sigma \equiv 1$	Simple polygon σ pos.	Simple polygon σ arb.	Polygon with holes $\sigma \equiv 1$	Polygon with holes σ pos.	Polygon with holes σ arb.
$\mathcal{S}(P)$ is connected	✓	✓	✓	✓	✓	×
$\mathcal{S}(P)$ has no crossing	✓	✓	×	✓	✓	×
$f(e)$ is monotone w.r.t. e	✓	×	×	✓	×	×
bd $f(e)$ is a simple polygon	✓	✓	×	✓	×	×
$\mathcal{T}(P)$ is z-monotone[1]	✓	✓	×	✓	✓	×
$\mathcal{S}(P)$ has $n(\mathcal{S}(P)) - 1 + h$ edges	✓	✓	×	✓	✓	×
$\mathcal{S}(P)$ is a tree	✓	✓	×			

Table: [Biedl et al., 2013, Biedl et al., 2015]

[1] $\mathcal{T}(P) := \bigcup_{t \geq 0} \mathcal{W}_P(t) \times \{t\}$.

Ambiguity of edge events

Ambiguity for parallel edges of different weights become adjacent.

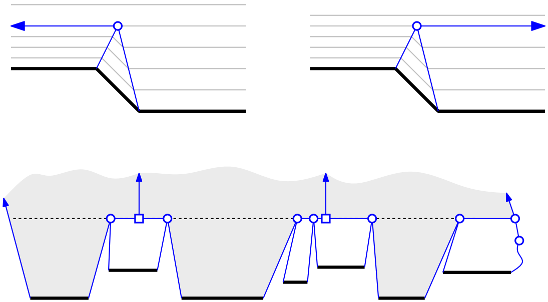
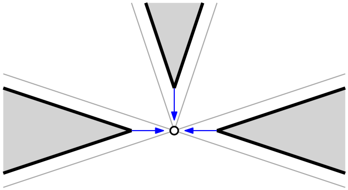


Figure: Resolution methods proposed in [Biedl et al., 2013, Biedl et al., 2015].

Split events: pairing edges

The *standard scheme* works for unweighted straight skeletons.



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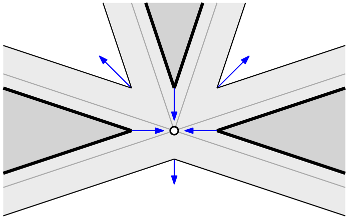
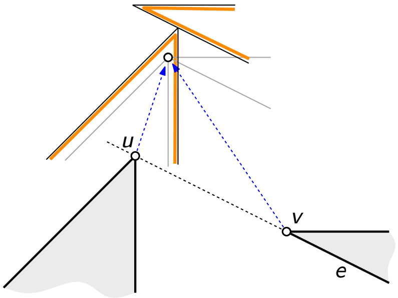


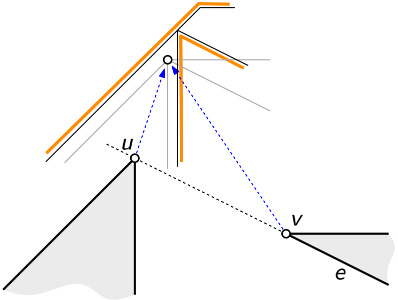
Fundamental principle: Between events, the wavefront is a planar collection of wavefront polygons.

Insufficiency of standard pairing technique

- No pairing may exist that gives a planar wavefront.
- The standard pairing technique can cause intersections.

Insufficiency of standard pairing technique





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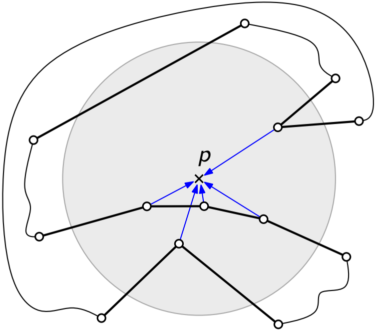
Split events with many vertices

How to handle this?



Weak planarity

- Fix a planar graph $G = (V, E)$.
- Denote by Φ all its straight-line embeddings $\varphi\colon V \to \mathbb{R}^2$.
  - Endow Φ with a metric
    $d(.,.)_\infty\colon \Phi \times \Phi \to \mathbb{R}, \quad d(\varphi, \varphi')_\infty = \max_{v \in V} \|\varphi(v) - \varphi'(v)\|$.
- Note that the set of planar straight-line embeddings is an open set.

Definition

The set of *weakly-planar* embeddings of G is the topological closure of the set of planar embeddings of G.

Corollary

For every weakly-planar embedding φ and for every $\varepsilon > 0$ there is a planar ε-perturbation φ' of φ, that is, $d(\varphi, \varphi')_\infty < \varepsilon$.

New fundamental principle

At all times, the wavefront is a weakly-planar collection of polygons.

Definition: Event

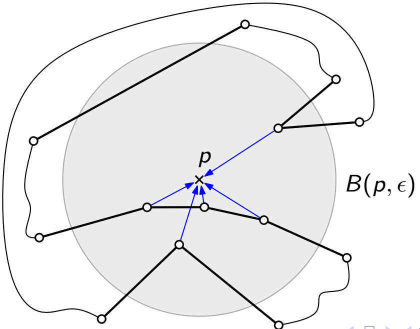
Definition

At location p and time t an *event* happens if $\exists \varepsilon_0 > 0 \; \forall 0 < \varepsilon < \varepsilon_0 \; \exists \delta > 0$ s.t.

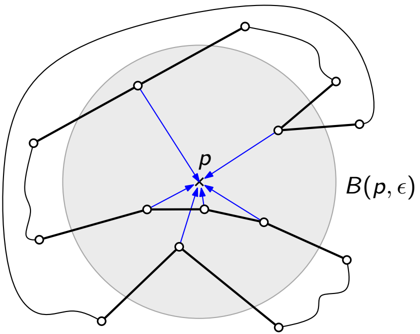
(i) $\mathcal{W}(t') \cap B(p, \varepsilon)$ is non-empty and weakly-planar for $t' \in [t - \delta, t]$ and

(ii) $\mathcal{W}(t') \cap B(p, \varepsilon)$ is non-empty and not weakly-planar for $t' \in (t, t + \delta]$

or if at least two vertices meet at time t at p. We call the edges that meet p at time t the edges that are *involved* in the event.



Pairing edges



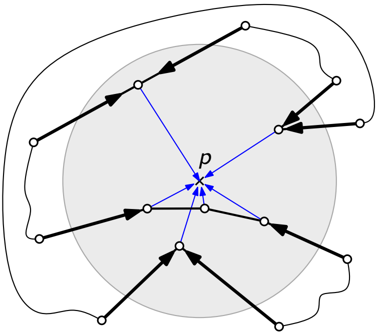
First:

- Remove collapsed edges.
- Split edges where both endpoints are outside $B(p, \varepsilon)$.

Task: Find a pairing of remaining edges to restore weak planarity.

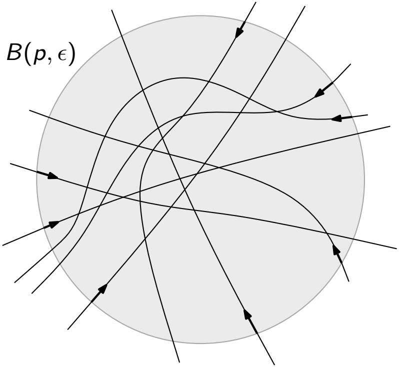
- Is this always possible? Uniquely?

Directed pseudo-line arrangements



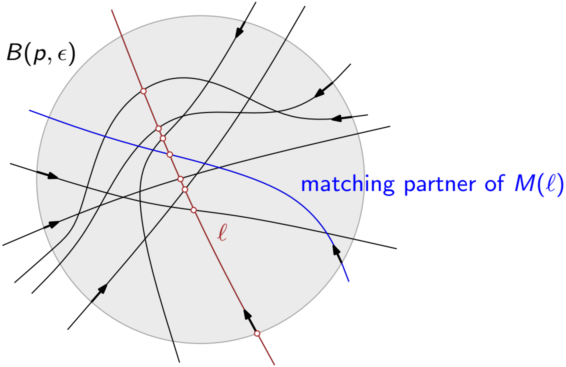
- We have k involved chains.
  - Hence, $2k$ (non-collapsed) edges.
  - Assign direction to each edge.
- Consider directed supporting lines of edges, after the event.
  - Perturb a little to obtain general position.
  - $\rightarrow$ pseudo-line arrangement $\mathcal{L}$ of directed pseudo-lines.
  - Obtain a "planar matching" of $\mathcal{L}$ and revert perturbation.

Planar matchings



- Given: even number of directed pseudo lines.
- Every pair intersects, in a single unique point, within $B(p, \epsilon)$.

Planar matchings



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- Matching: grouping into pairs.

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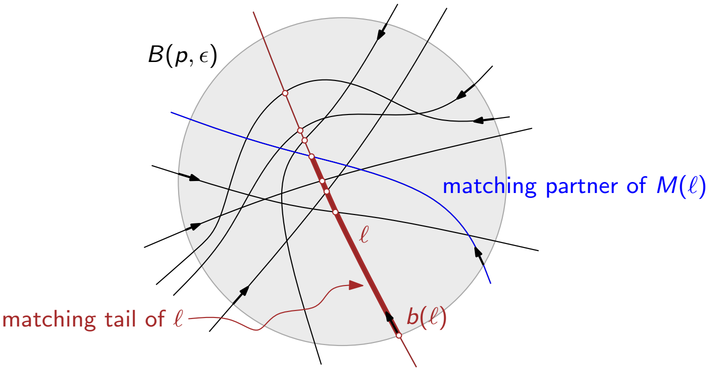

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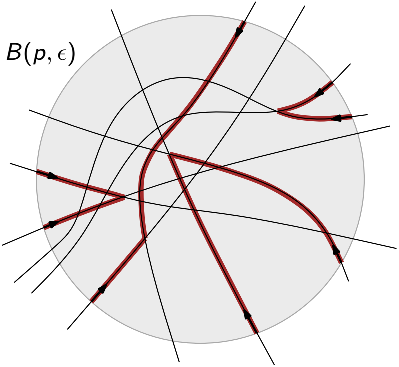
- Matching: grouping into pairs.
- Planar matching: matching tails do not cross.

Planar matchings



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Planar matchings

Theorem

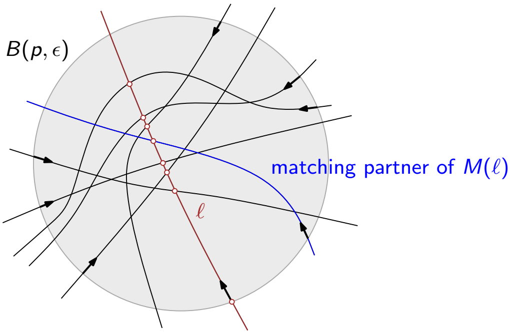
Every directed pseudo-line arrangement has a planar matching.

Stable roommates and planar matchings

- We are given a directed pseudo-line arrangement $\mathcal{L}$.
  - Intersection order gives the ranking.
- A matching is stable if:
  - There is no pair s.t. they prefer each other over their matching partners.

Lemma

$\mathcal{L}$ has a planar matching if and only if there is a stable matching.



Stable partitions

- A partition is a permutation π of $\ell_1, \dots, \ell_N$.
  - Decompose π into cycles.
- A partition is stable if:
  - For all cylces size ≥ 3: each ℓ prefers $\pi(\ell)$ over $\pi^{-1}(\ell)$.
  - There is no pair $\{\ell_i, \ell_j\}$ s.t. they prefer each other over $\pi^{-1}(\ell_i)$ and $\pi^{-1}(\ell_j)$

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Theorem ([Tan and Hsueh, 1995])

1. *There is a stable partition, and it can be found in polynomial time.*
2. *There is a stable matching if and only if there is a stable partition with no cycles of odd size.*

Theorem

There are no cycles of odd size for directed pseudo-line arrangements.

Odd parties do not exist

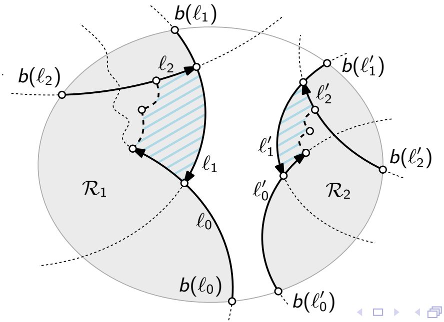
Party-tail of ℓ: part between $b(\ell)$ and $\ell \times \pi^{-1}(\ell)$ (or endpoint for $\ell = \pi(\ell)$).

Lemma

The party-tails of ℓ and ℓ' do not intersect, unless $\pi(\ell) = \ell'$ or $\pi(\ell') = \ell$.

Lemma

- *There cannot be two parties of size at least three.*
- *Singletons do not exist.*



Existence of planar matchings

Theorem

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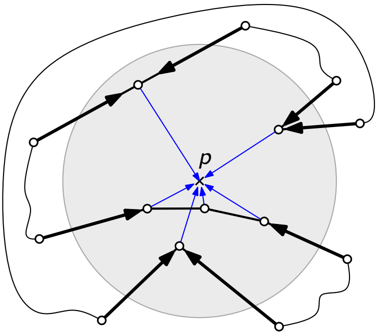

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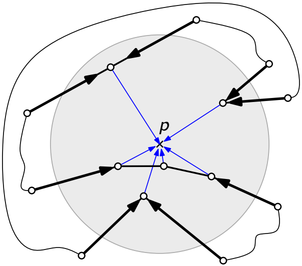

- Perturb directed edges such that
  - Edges still reach p at time t.
  - Supporting lines at $t + \delta$ are in general position.
  - Perturbed $\mathcal{W}$ is strictly-planar outside $B(p, \varepsilon)$.
  - Vertices do not jump over supporting lines.
- Compute planar matching.
- Revert perturbation.

Lemma

The new post-event wavefront is weakly-planar.

Summary

Prior status quo:

- Weighted straight skeletons are half-established:
  - Algorithms, implementations, theory & practice.
- Lack of solid foundation.

Contribution:

- Unified, generalized definition of events.
  - Maybe interesting for higher dimensions too.
- First rigorous definition of weighted straight skeletons.
- We prove that event handling can be always done, i.e., the weighted straight skeleton actually always exists.
- Planar matchings of directed pseudo-line arrangements might be interesting for their own.

Acknowledgement: We thank David Eppstein for suggesting the idea of using stable matchings.

Bibliography I

Aichholzer, O., Alberts, D., Aurenhammer, F., and Gärtner, B. (1995).
A novel type of skeleton for polygons.
J. Universal Comp. Sci., 1(12):752–761.

Biedl, T., Held, M., Huber, S., Kaaser, D., and Palfrader, P. (2013).
Weighted straight skeletons in the plane.
In *Proc. 25th Canad. Conf. on Comp. Geom. (CCCG '13)*, pages 13–18, Waterloo, Canada.

Biedl, T., Held, M., Huber, S., Kaaser, D., and Palfrader, P. (2015).
Weighted straight skeletons in the plane.
Comp. Geom. Theory & Appl., 48(2):120–133.

Eppstein, D. and Erickson, J. (1999).
Raising roofs, crashing cycles, and playing pool: Applications of a data structure for finding pairwise interactions.
Discrete Comp. Geom., 22(4):569–592.

Tan, J. J. and Hsueh, Y.-C. (1995).
A generalization of the stable matching problem.
Discrete Applied Mathematics, 59(1):87–102.

Non-uniqueness