

# Recognizing Straight Skeletons and Voronoi Diagrams and Reconstructing Their Input

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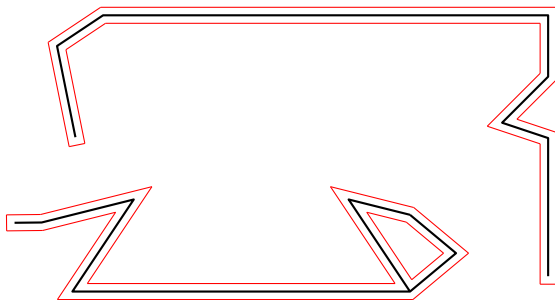
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July 8–10

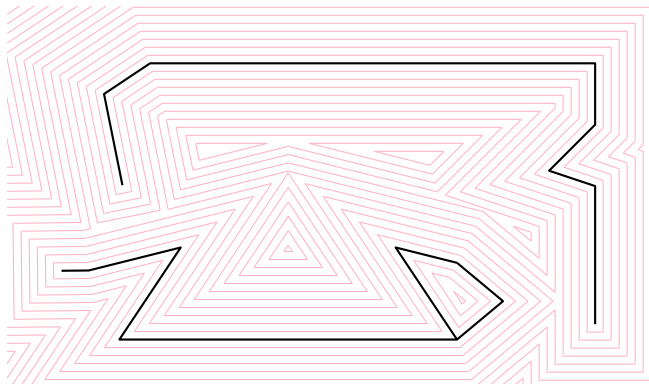
# Straight skeleton of a PSLG

- ▶ [Aichholzer and Aurenhammer, 1998]: straight skeleton  $\mathcal{S}(G)$  of a PSLG  $G$



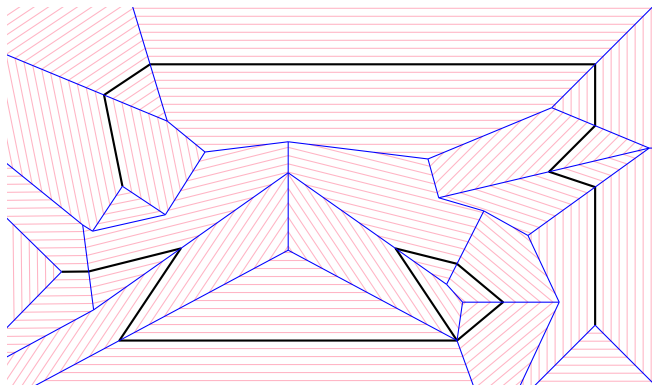
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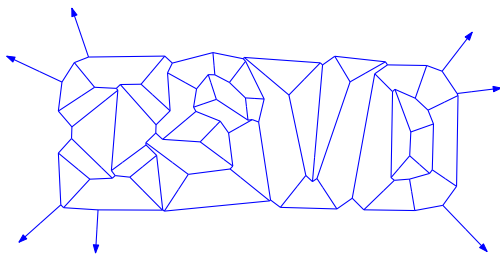


# Problem statement

PSLG<sup>∞</sup>: edges may be straight-line segments or rays.

## Problem (GMP-SS)

Given a PSLG<sup>∞</sup>  $G$ , can we find a PSLG  $H$  such that  $\mathcal{S}(H) = G$ ?

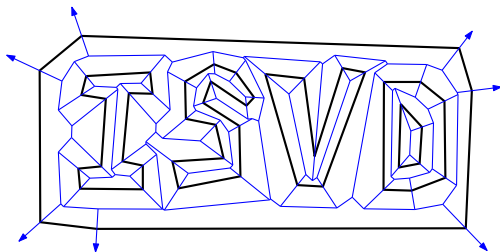


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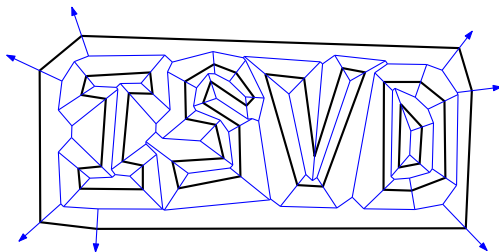


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## Problem [Aichholzer et al., 1995]

Give necessary and sufficient conditions for  $G$  to be the straight skeleton of  $H$ .

# Prior work

[Aichholzer et al., 2012]:

- ▶ Any **abstract tree**  $T$  can be realized as  $\mathcal{S}(P)$  (or  $\mathcal{V}(P)$ ) of a convex polygon.
- ▶ Realizability of **phylogenetic trees**  $T$  as  $\mathcal{S}(P)$  of a polygon  $P$ .



# Outline

## Part I: Straight skeletons

- ▶ Characterization of straight skeletons.
  - ▶ Three necessary and sufficient conditions for  $G$  to be the straight skeleton of a specific input.
- ▶ Recognizing straight skeletons.
  - ▶ How to determine whether  $G$  is the straight skeleton of some input?
- ▶ Reconstruction algorithm.
  - ▶ How to compute the input?

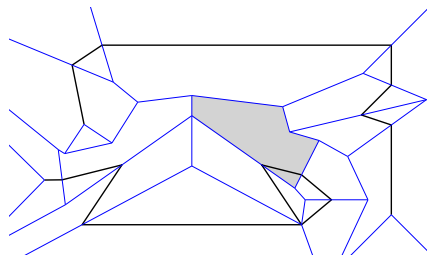
## Part II: Voronoi diagrams

- ▶ The framework developed in Part I can be applied to Voronoi diagrams.

# Characterization: basic facts

## Facts

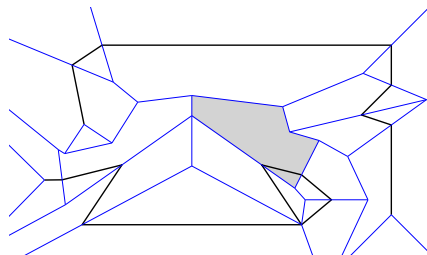
- ▶ Every edge of  $\mathcal{S}(H)$  is on the bisector of two edges of  $H$ .
- ▶ Every face of  $\mathcal{S}(H)$  contains exactly one segment of  $H$ , except for faces generated by degree-one vertices of  $H$ .
- ▶ Every edge of  $H$  begins and ends at an edge of  $\mathcal{S}(H)$ .
- ▶ If a vertex of  $\mathcal{S}(H)$  has degree two then it coincides with a degree-one vertex of  $H$ . All other vertices have degree three or higher.



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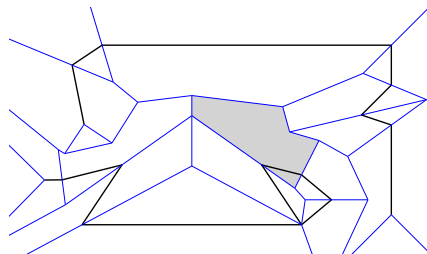
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**Temporary assumption:**  $G$  has no degree-2 vertices.

## Characterization: inside-condition

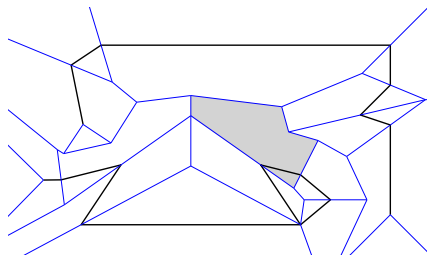
Let  $G$  be the putative straight skeleton and  $F$  the set of faces of  $G$ .



A solution to GMP-SS can be denoted as a mapping  $\lambda: F \rightarrow \mathcal{L}$ , where  $\mathcal{L}$  is the set of lines.

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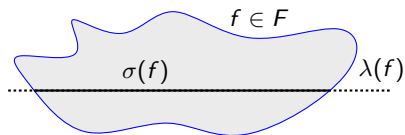
A solution to GMP-SS can be denoted as a mapping  $\lambda: F \rightarrow \mathcal{L}$ , where  $\mathcal{L}$  is the set of lines.

### Definition (Inside-condition)

$\lambda$  fulfills the inside-condition if  $\sigma(f) := \lambda(f) \cap f$  is a **single line segment** for all  $f \in F$ .

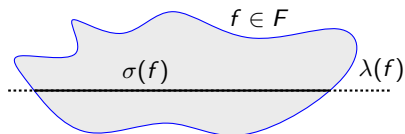
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We construct  $H$  as the graph whose edges are  $\sigma(f)$ , with  $f \in F$ .



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For a  $G$  and  $\lambda$  we denote by  $G^* := G \cup H$  and by  $F^*$  the faces of  $G^*$ .

- ▶ Every face of  $G$  contains two faces of  $G^*$ .
- ▶ We reuse  $\lambda(f)$  and  $\sigma(f)$  for faces of  $G^*$  accordingly.

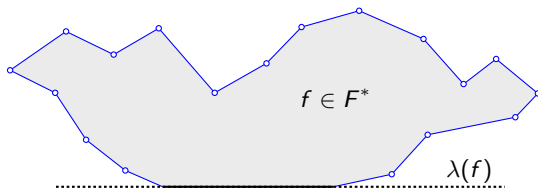
# Characterization: sweeping-condition

## Definition (Sweeping-condition)

A face  $f$  of  $G^*$  fulfills the sweeping-condition if

1.  $f$  is **monotone** w.r.t.  $\lambda(f)$  and
2. at the lower chain, the **distance to  $\lambda(f)$  is increasing**, when moving away from  $\sigma(f)$ .

$\lambda$  fulfills the sweeping-condition if all faces of  $G^*$  fulfill it.

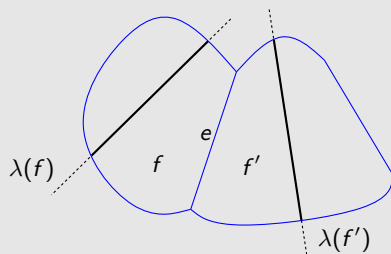




# Characterization: bisector-condition

## Definition (Bisector-condition)

The edge  $e = f \cap f'$  fulfills the bisector-condition if  $e$  lies on the bisector of  $\lambda(f)$  and  $\lambda(f')$ .



$\lambda$  fulfills the bisector-condition if all edges of  $G$  fulfill the bisector-condition.

# Characterization

## Lemma

*If  $\lambda$  solves GMP-SS then  $\lambda$  fulfills the inside-, sweeping-, and bisector-condition.*

*Proof.* Inside- and bisector-condition: by definition of straight skeletons.

Sweeping-condition:

- ▶ Monotonicity by [Aichholzer et al., 1995].
- ▶ Lower chain is even convex by [Huber, 2012].



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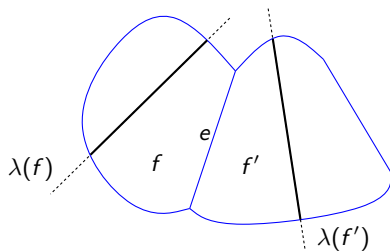
## Theorem

*If  $\lambda$  fulfills the inside-, sweeping-, and bisector-condition then  $\lambda$  solves GMP-SS.*

# Recognizing straight skeletons

**Key method:** We successively reflect lines  $\lambda(f)$  at edges of  $f$ .

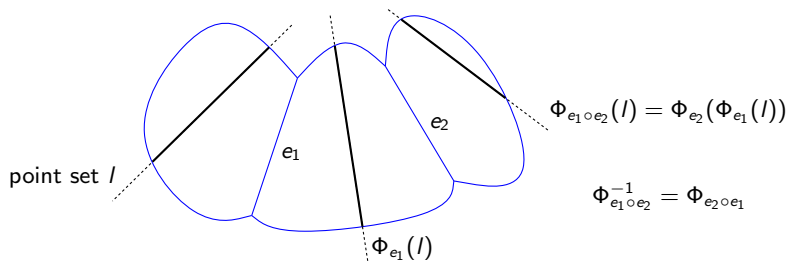
- ▶ Assume we know a suitable  $\lambda(f)$  for one face  $f$ .
- ▶ Bisector-condition: we know  $\lambda(f')$  for a neighboring face  $f'$ , too.
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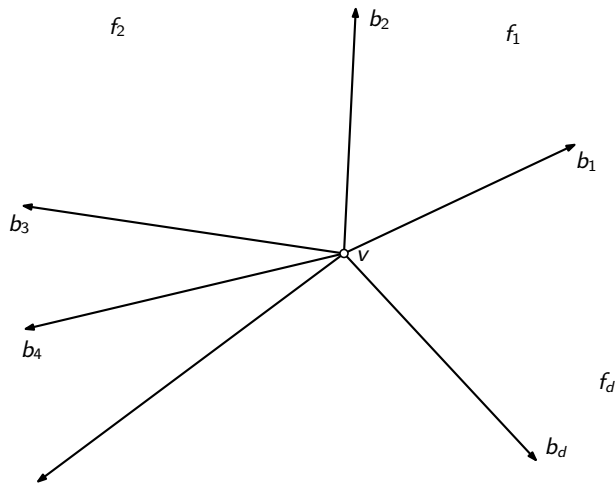
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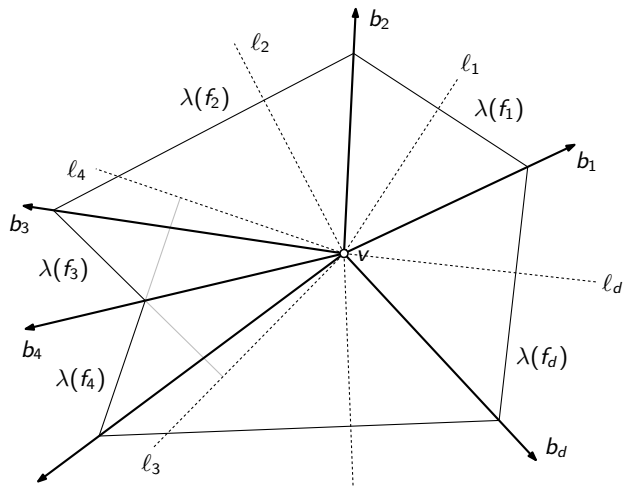
# Recognizing straight skeletons: star graphs

- ▶ “Local view” at a vertex  $v$  of  $G$  with incident ray-edges  $b_1, \dots, b_d$ .
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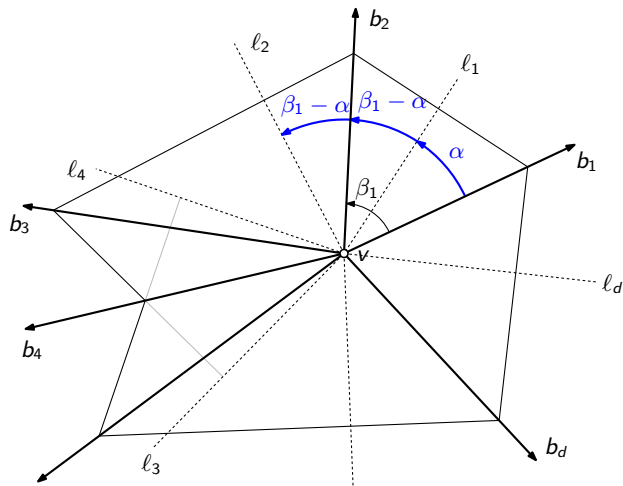
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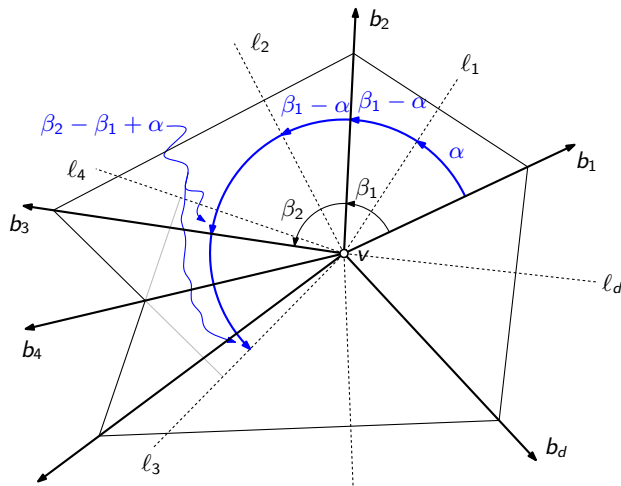
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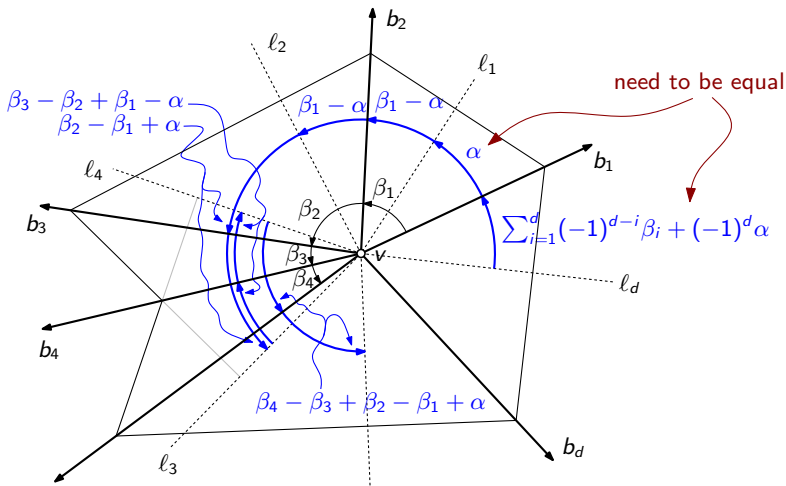
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We get  $\alpha = \sum_{i=1}^d (-1)^{d-i} \beta_i + (-1)^d \alpha$  and therefore

$$\frac{1}{2} \sum_{i=1}^d (-1)^{d-i} \beta_i = \begin{cases} 0 & \text{if } d \text{ is even,} \\ \alpha & \text{if } d \text{ is odd.} \end{cases} \quad (1)$$

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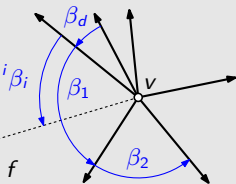
The vertex  $v$  with even degree  $d$  fulfills the **balance-condition** if

$$\beta_d - \beta_{d-1} + \dots + \beta_2 - \beta_1 = 0.$$

For vertices of odd degree  $d$  we define  $\ell(f, v)$  as

$$\frac{1}{2} \sum_{i=1}^d (-1)^{d-i} \beta_i$$

$$\ell(f, v) :=$$

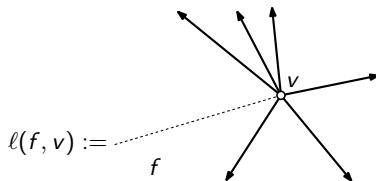


# Recognizing straight skeletons: star graphs

## Lemma

$\Phi_{b_1 \circ \dots \circ b_d}(l) = l$  if and only if

$$\begin{cases} v \text{ fulfills the balance-condition} & \text{if } d \text{ is even} \\ l = \ell(f, v) \vee l \perp \ell(f, v) \text{ for some } f \in F \text{ with } v \in f & \text{if } d \text{ is odd} \end{cases} \quad (2)$$



# Recognizing straight skeletons: PSLGs

The previous lemma imposes constraints on  $\lambda$  for the vertices of  $G$ :

$$\ell(f) := \{l \in \mathcal{L} : l \cap \text{int } f \neq \emptyset\} \cap \bigcap_{\substack{v \text{ is vertex of } f \\ \deg(v) \text{ is odd}}} \{\ell(f, v)\} \cup \ell(f, v)^\perp. \quad (3)$$

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- ▶ Choose a spanning tree  $T$  of the dual of  $G$ , with a root face  $r$ .
- ▶ Denote by  $f \rightsquigarrow^T r$  the sequence of edges in  $T$  from  $f$  to  $r$  and define

$$f^r := \Phi_{f \rightsquigarrow^T r}(f) \quad (4)$$

$$\ell^r(f) := \Phi_{f \rightsquigarrow^T r}(\ell(f)) \quad (5)$$

$$X := \bigcap_{f \in \mathcal{F}} \ell^r(f). \quad (6)$$

# Recognizing straight skeletons: PSLGs

## Theorem

*GMP-SS for  $G$  has a solution if and only if*

- ▶ *the balance-condition holds for all vertices of even degree and*
- ▶ *there is a line  $l \in X$  such that for all  $f \in F$* 
  - ▶  *$l \cap f^r$  is a single segment and*
  - ▶ *the components of  $f^r \setminus l$  fulfill the sweeping-condition.*

*There is a one-to-one correspondence between such lines  $l \in X$  and solutions to GMP-SS.*

# Recognizing straight skeletons: PSLGs

Proof sketch:

- ▶ Take a suitable  $I$  and define  $\lambda(f) := \Phi_{r \rightsquigarrow T_f}(I)$ .
- ▶ To show:  $\lambda$  fulfills the inside-, bisector- and sweeping-condition.
  - ▶ Inside- and sweeping-condition are fulfilled by assumption.
  - ▶ Bisector-condition for (duals of) edges in  $T$  as well.

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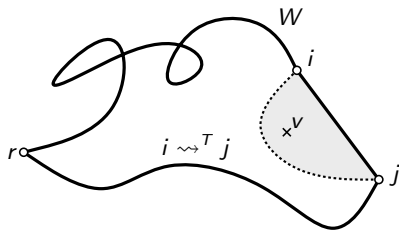
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- ▶ **Claim:** edges not in  $T$  fulfill the bisector-condition as well.
- ▶ **Stronger claim:** Let  $W$  be any walk in the dual of  $G$  from  $r$  to  $j$ . Then  $\Phi_W(\lambda(r)) = \lambda(j)$ . That is, it does not matter how we choose  $T$ .



## Reconstructing the input: algorithm

We are given  $G$  and want to find a suitable  $\lambda$ , i.e., a suitable  $l \in X$ .

- ▶ Check that balance-condition holds at every even-degree vertex.
- ▶ We compute  $T$ , all  $f^r = \Phi_{f \rightsquigarrow T_r}(f)$  and all  $l^r(f, v) = \Phi_{f \rightsquigarrow T_r}(l(f, r))$  in total linear time.

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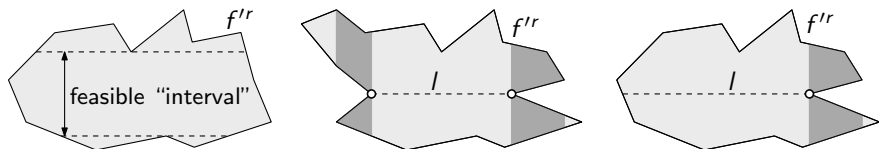
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- ▶ **Case 1:** All vertices have even degree.
  - ▶ By the balance-condition all faces are convex.
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  - ▶ Using [Edelsbrunner et al., 1989] and [Hershberger, 1989] we find all lines  $l$  traversing all int  $f^r$  in  $O(n \log n)$  time.

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- ▶ **Case 2:** At least one vertex  $v$  has odd degree.
  - ▶ A suitable  $l$  has fixed direction: identical or perpendicular to  $\ell^r(f, v)$ .
  - ▶ inside/sweeping condition  $\Rightarrow$  restrict all suitable  $l$  to an “interval” of parallel lines in  $O(n)$  time.





# Reconstructing the input

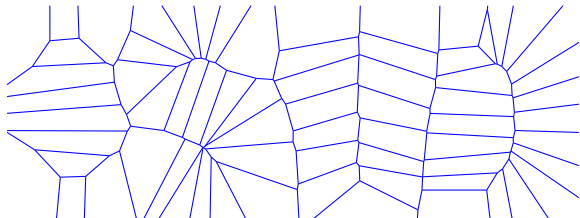
## Theorem

*GMP-SS can be solved and the set of solutions can be found in  $O(n \log n)$  time of a  $PSLG^\infty$   $G$  with  $n$  edges.*

# Voronoi diagrams

## Problem (GMP-VD)

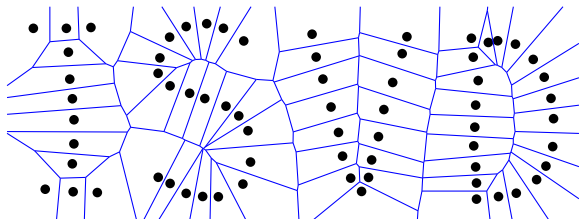
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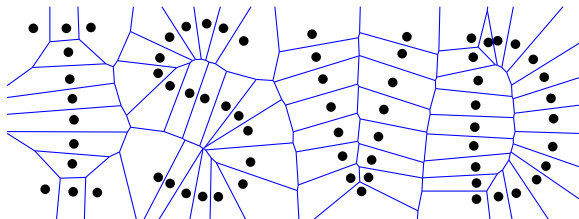
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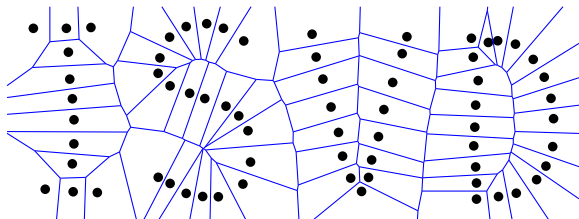
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### Prior work:

- ▶ [Ash and Bolker, 1985]: Solve GMP-VD if all vertices have **odd degree**.
- ▶ [Hartvigsen, 1992]: Solve GMP-VD by means of linear programming.

# Characterization of Voronoi diagrams

We denote a solution of GMP-VD as a mapping  $\rho: F \rightarrow \mathbb{R}^2$ .

- ▶ We look for  $\rho$  such that  $\mathcal{V}(\{\rho(f): f \in F\}) = G$ .

## Lemma ([Ash and Bolker, 1985])

$\rho$  solves GMP-VD if

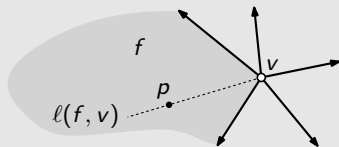
- ▶ **Inside-condition:**  $\rho(f) \in \text{int } f$  for all  $f \in F$ .
- ▶ **Bisector-condition:**  $e$  is on the bisector of  $\rho(f), \rho(f')$  for any edge  $e = f \cap f'$ .

# Recognizing Voronoi diagrams

## Lemma

$\Phi_{b_1 \circ \dots \circ b_d}(p) = p$  if and only if

$$\begin{cases} v \text{ fulfills the balance-condition} & \text{if } d \text{ is even} \\ p \in \ell(f, v) \text{ for some } f \in F \text{ with } v \in f & \text{if } d \text{ is odd} \end{cases} \quad (7)$$

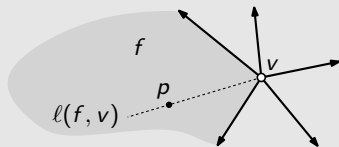


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We again define

$$S(f) := (\text{int } f) \cap \bigcap_{\substack{v \text{ is vertex of } f \\ \deg(v) \text{ is odd}}} \ell(f, v) \quad \text{Easily becomes a single point.} \quad (8)$$

$$X := \bigcap_{f \in F} \Phi_{f \rightsquigarrow T_r}(S(f)) \quad \text{Every point implies a solution } \rho. \quad (9)$$



# Conclusion

Characterization of straight skeletons:

- ▶ Deeper insight in the geometry and structure of  $\mathcal{S}(H)$ .
- ▶ Allows for necessary and sufficient  $O(n)$  time a-posteriori checks of the validity of  $\mathcal{S}(H)$  in straight-skeleton codes.

We solve GMP-SS and GMP-VD on  $G$

- ▶ using a **unified framework** based on reflections on edges of a spanning tree of the dual of  $G$
- ▶ in  $O(n \log n)$  time.
- ▶ First result on GMP-SS.
- ▶ Closes a gap in [Ash and Bolker, 1985] for GMP-VD when vertices have even degree.

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# Characterization: Proof

**Key idea:**  $G$  and  $\mathcal{S}(H)$  each impose a wavefront-propagation process,  $\mathcal{W}_G(t)$  and  $\mathcal{W}_{\mathcal{S}(H)}(t)$ .

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## Lemma

*The initial wavefronts  $\mathcal{W}_G(\epsilon)$  and  $\mathcal{W}_{\mathcal{S}(H)}(\epsilon)$  are identical.*

# Characterization: Proof

## Lemma

Assume that  $\mathcal{W}_G(t') = \mathcal{W}_{S(H)}(t')$  for  $0 < t' < t$ .

- ▶ If  $\mathcal{W}_G(t)$  hits a vertex  $v$  of  $G^*$ , then  $v$  coincides with a vertex of  $S(H)$ .
- ▶ Analogously for  $\mathcal{W}_{S(H)}$ .



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## Theorem

$\mathcal{W}_G(t) = \mathcal{W}_{S(H)}(t)$  for all  $t$ .

*Proof.* [Sketch]

- ▶ By induction on the chronological order when  $\mathcal{W}_G$  resp.  $\mathcal{W}_{S(H)}$  hits a vertex  $v$  of  $G$  resp.  $S(H)$ .
- ▶ In a neighborhood of  $v$  we have swept and not-yet-swept cones.
- ▶ Insight: In the not-yet-swept cones contain each exactly one “outgoing” edge of  $G$  resp.  $S(H)$ .
- ▶ Claim: these edges are identical in the neighborhood of  $v$ .

# Non-unique solutions to GMP-SS and GMP-VD

