Recognizing Straight Skeletons and Voronoi Diagrams and Reconstructing Their Input

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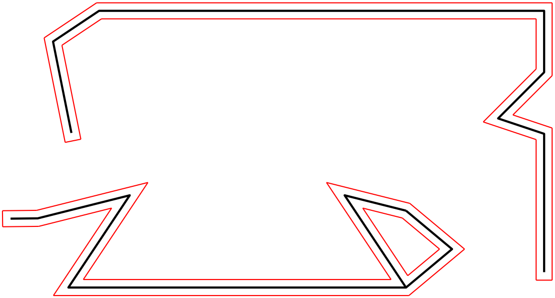
Straight skeleton of a PSLG

- [Aichholzer and Aurenhammer, 1998]: straight skeleton $\mathcal{S}(G)$ of a PSLG G



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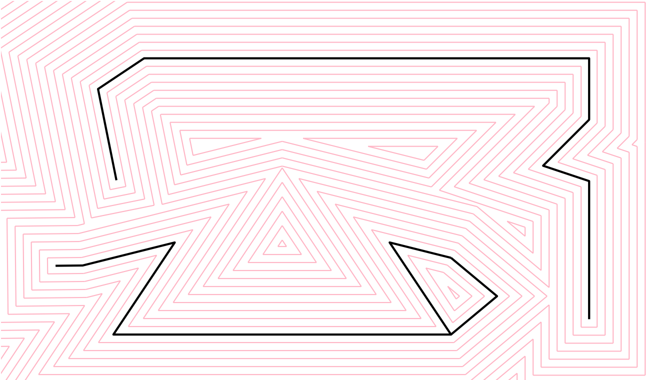

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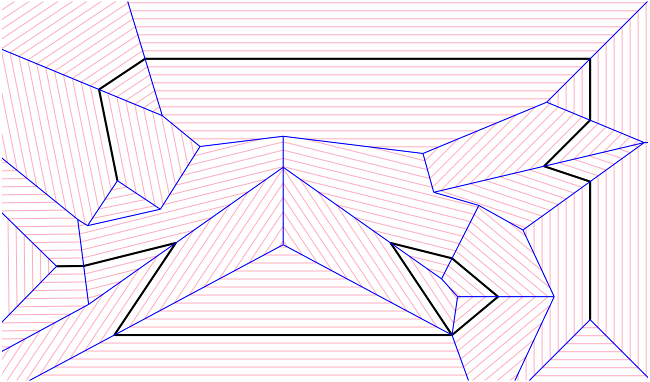
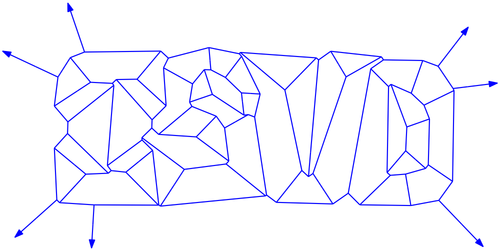


Problem statement

PSLG$^\infty$: edges may be straight-line segments or rays.

Problem (GMP-SS)

Given a PSLG$^\infty$ G, can we find a PSLG H such that $\mathcal{S}(H) = G$?



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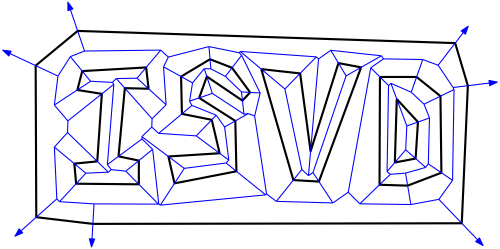

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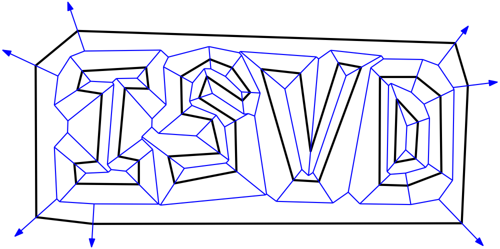


Problem [Aichholzer et al., 1995]

Give necessary and sufficient conditions for G to be the straight skeleton of H.

Prior work

[Aichholzer et al., 2012]:

- Any abstract tree T can be realized as $\mathcal{S}(P)$ (or $\mathcal{V}(P)$) of a convex polygon.
- Realizability of phylogenetic trees T as $\mathcal{S}(P)$ of a polygon P.

Outline

Part I: Straight skeletons

- Characterization of straight skeletons.
  - Three necessary and sufficient conditions for G to be the straight skeleton of a specific input.
- Recognizing straight skeletons.
  - How to determine whether G is the straight skeleton of some input?
- Reconstruction algorithm.
  - How to compute the input?

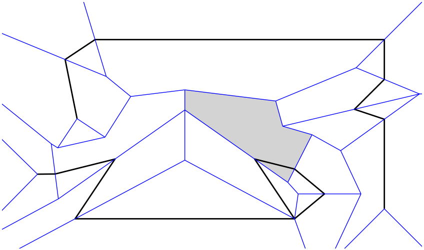
Part II: Voronoi diagrams

- The framework developed in Part I can be applied to Voronoi diagrams.

Characterization: basic facts

Facts

- Every edge of $\mathcal{S}(H)$ is on the bisector of two edges of H.
- Every face of $\mathcal{S}(H)$ contains exactly one segment of H, except for faces generated by degree-one vertices of H.
- Every edge of H begins and ends at an edge of $\mathcal{S}(H)$.
- If a vertex of $\mathcal{S}(H)$ has degree two then it coincides with a degree-one vertex of H. All other vertices have degree three or higher.



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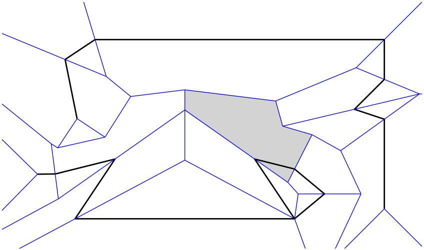
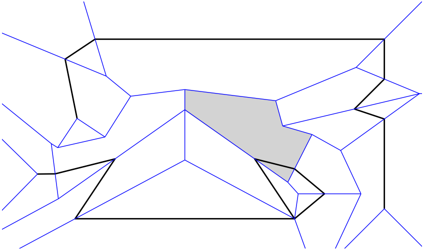


Temporary assumption: G has no degree-2 vertices.

Characterization: inside-condition

Let G be the putative straight skeleton and F the set of faces of G.



A solution to GMP-SS can be denoted as a mapping $\lambda \colon F \to \mathcal{L}$, where $\mathcal{L}$ is the set of lines.

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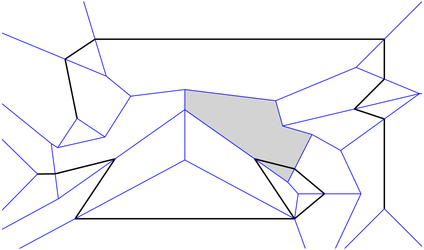


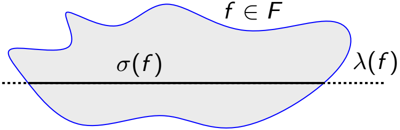
A solution to GMP-SS can be denoted as a mapping $\lambda \colon F \to \mathcal{L}$, where $\mathcal{L}$ is the set of lines.

Definition (Inside-condition)

λ fulfills the inside-condition if $\sigma(f) := \lambda(f) \cap f$ is a single line segment for all $f \in F$.

Characterization: inside-condition

We construct H as the graph whose edges are $\sigma(f)$, with $f \in F$.



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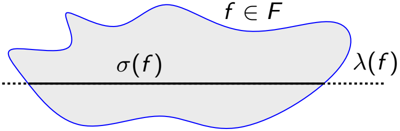


For a G and λ we denote by $G^* := G \cup H$ and by F^* the faces of G^*.

- Every face of G contains two faces of G^*.
- We reuse $\lambda(f)$ and $\sigma(f)$ for faces of G^* accordingly.

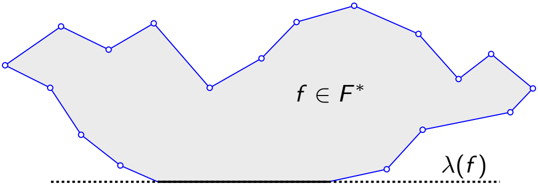
Characterization: sweeping-condition

Definition (Sweeping-condition)

A face f of G^* fulfills the sweeping-condition if

1. f is **monotone** w.r.t. $\lambda(f)$ and
2. at the lower chain, the **distance to** $\lambda(f)$ **is increasing**, when moving away from $\sigma(f)$.

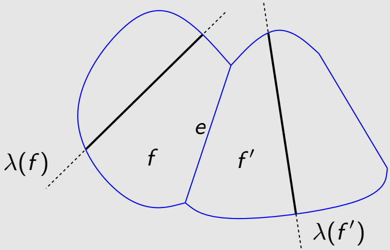
λ fulfills the sweeping-condition if all faces of G^* fulfill it.



Characterization: bisector-condition

Definition (Bisector-condition)

The edge $e = f \cap f'$ fulfills the bisector-condition if e lies on the bisector of $\lambda(f)$ and $\lambda(f')$.



λ fulfills the bisector-condition if all edges of G fulfill the bisector-condition.

Characterization

Lemma

If λ solves GMP-SS then λ fulfills the inside-, sweeping-, and bisector-condition.

Proof. Inside- and bisector-condition: by definition of straight skeletons.
Sweeping-condition:

- Monotonicity by [Aichholzer et al., 1995].
- Lower chain is even convex by [Huber, 2012].

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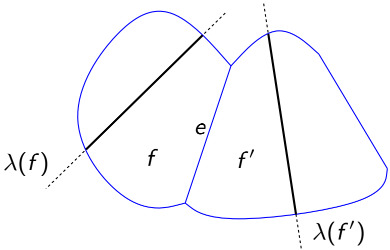
Theorem

If λ fulfills the inside-, sweeping-, and bisector-condition then λ solves GMP-SS.

Recognizing straight skeletons

Key method: We successively reflect lines $\lambda(f)$ at edges of f.

- Assume we know a suitable $\lambda(f)$ for one face f.
- Bisector-condition: we know $\lambda(f')$ for a neighboring face f', too.
- Going along a spanning tree of the dual of G, we know $\lambda(f')$ for all $f' \in F$.



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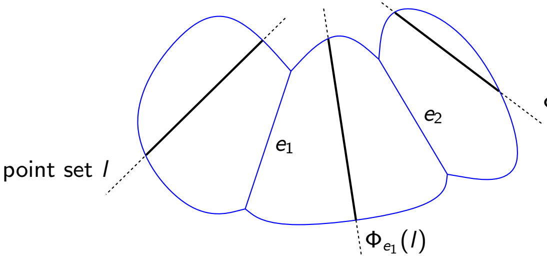
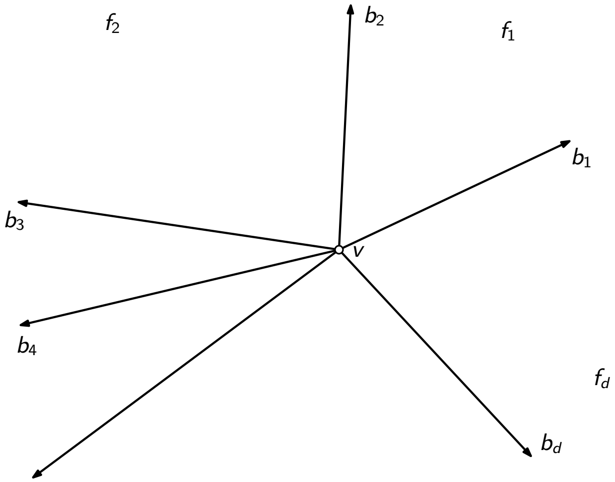


$$\Phi_{e_1 \circ e_2}(l) = \Phi_{e_2}(\Phi_{e_1}(l))$$

$$\Phi^{-1}_{e_1 \circ e_2} = \Phi_{e_2 \circ e_1}$$

Recognizing straight skeletons: star graphs

- "Local view" at a vertex v of G with incident ray-edges $b_1, \ldots, b_d$.
  - Find λ that fulfills inside-, (sweeping-), and bisector-condition.



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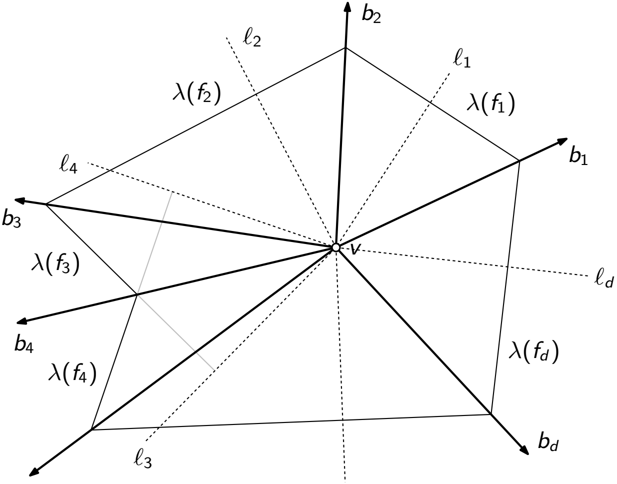

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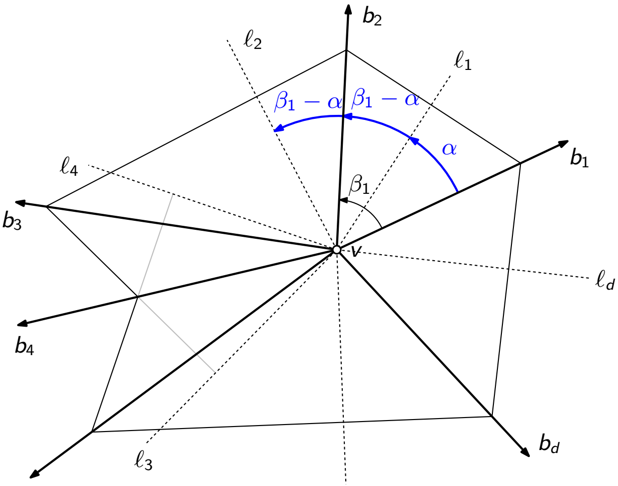

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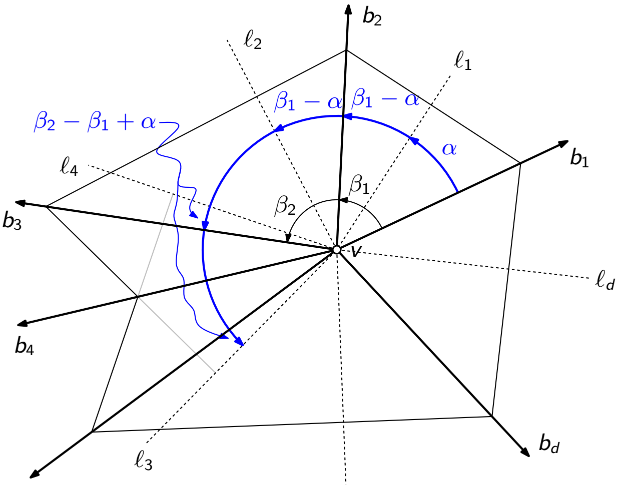

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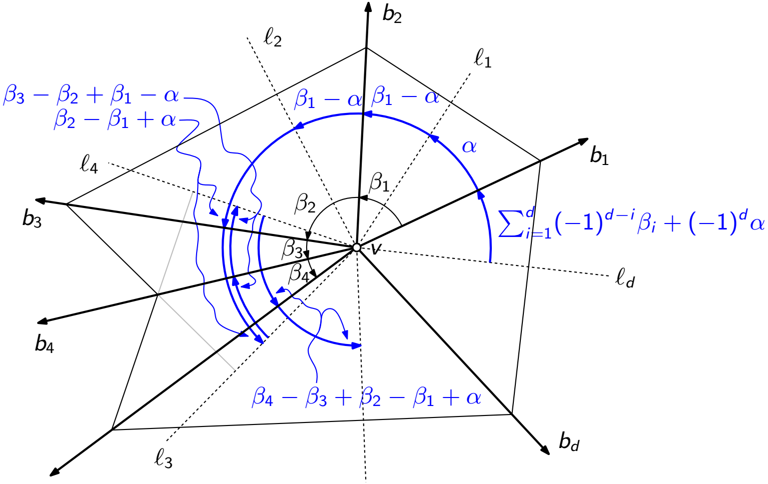

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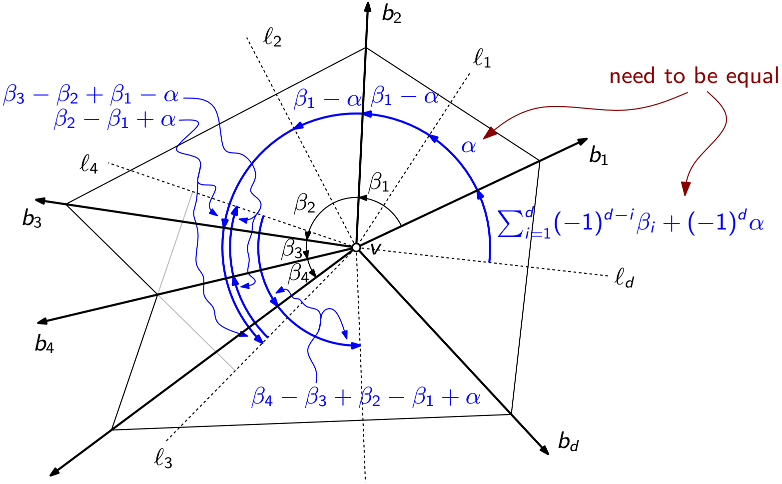

Recognizing straight skeletons: star graphs

We get $\alpha = \sum_{i=1}^{d} (-1)^{d-i} \beta_i + (-1)^d \alpha$ and therefore

$$\frac{1}{2} \sum_{i=1}^{d} (-1)^{d-i} \beta_i = \begin{cases} 0 & \text{if } d \text{ is even,} \\ \alpha & \text{if } d \text{ is odd.} \end{cases} \tag{1}$$

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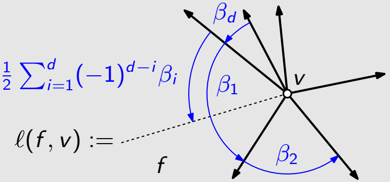
Definition

The vertex v with even degree d fulfills the **balance-condition** if $\beta_d - \beta_{d-1} + \cdots + \beta_2 - \beta_1 = 0$.

For vertices of odd degree d we define $\ell(f, v)$ as

$\ell(f,v) :=$

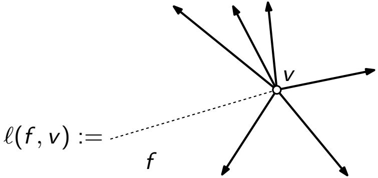


Recognizing straight skeletons: star graphs

Lemma

$\Phi_{b_1 \circ \cdots \circ b_d}(l) = l$ *if and only if*

$$\begin{cases} v \text{ fulfills the balance-condition} & \text{if } d \text{ is even} \\ l = \ell(f,v) \ \vee \ l \perp \ell(f,v) \text{ for some } f \in F \text{ with } v \in f & \text{if } d \text{ is odd} \end{cases} \quad (2)$$



Recognizing straight skeletons: PSLGs

The previous lemma imposes constraints on λ for the vertices of G:

$$\ell(f) := \{l \in \mathcal{L} : \; l \cap \operatorname{int} f \neq \emptyset\} \; \cap \bigcap_{\substack{v \text{ is vertex of } f \\ \deg(v) \text{ is odd}}} \{\ell(f,v)\} \cup \ell(f,v)^{\perp}. \tag{3}$$

We propagate the per-face constraints to a single face:

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- Choose a spanning tree T of the dual of G, with a root face r.

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We propagate the per-face constraints to a single face:

- Choose a spanning tree T of the dual of G, with a root face r.
- Denote by $f \rightsquigarrow^T r$ the sequence of edges in T from f to r and define

$$f^r := \Phi_{f \rightsquigarrow^T r}(f) \tag{4}$$

$$\ell^r(f) := \Phi_{f \rightsquigarrow^T r}(\ell(f)) \tag{5}$$

$$X := \bigcap_{f \in F} \ell^r(f). \tag{6}$$

Recognizing straight skeletons: PSLGs

Theorem

GMP-SS for G has a solution if and only if

- *the balance-condition holds for all vertices of even degree and*
- *there is a line $l \in X$ such that for all $f \in F$*
  - *$l \cap f^r$ is a single segment and*
  - *the components of $f^r \setminus l$ fulfill the sweeping-condition.*

There is a one-to-one correspondence between such lines $l \in X$ and solutions to GMP-SS.

Recognizing straight skeletons: PSLGs

Proof sketch:

- Take a suitable l and define $\lambda(f) := \Phi_{r \rightsquigarrow_T f}(l)$.
- To show: λ fulfills the inside-, bisector- and sweeping-condition.
  - Inside- and sweeping-condition are fulfilled by assumption.
  - Bisector-condition for (duals of) edges in T as well.

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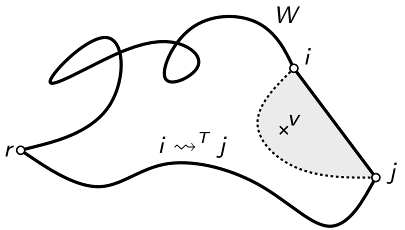
  - Inside- and sweeping-condition are fulfilled by assumption.
  - Bisector-condition for (duals of) edges in T as well.
- **Claim:** edges not in T fulfill the bisector-condition as well.
- **Stronger claim:** Let W be any walk in the dual of G from r to j. Then $\Phi_W(\lambda(r)) = \lambda(j)$. That is, it does not matter how we choose T.



Reconstructing the input: algorithm

We are given G and want to find a suitable λ, i.e., a suitable $I \in X$.

- Check that balance-condition holds at every even-degree vertex.
- We compute T, all $f^r = \Phi_{f \rightsquigarrow T_r}(f)$ and all $\ell^r(f, v) = \Phi_{f \rightsquigarrow T_r}(\ell(f, r))$ in total linear time.

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- **Case 1:** All vertices have even degree.
  - By the balance-condition all faces are convex.
    - Sweeping-condition is trivial.
  - Using [Edelsbrunner et al., 1989] and [Hershberger, 1989] we find all lines l traversing all int f^r in $O(n \log n)$ time.

Reconstructing the input: algorithm

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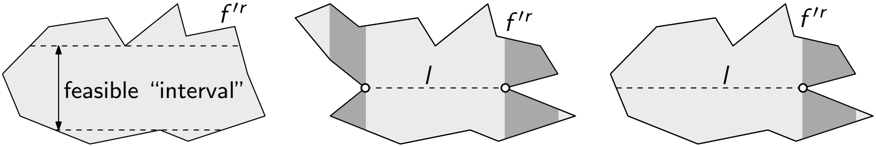
- **Case 2:** At least one vertex v has odd degree.
  - A suitable l has fixed direction: identical or perpendicular to $\ell^r(f, v)$.
  - inside/sweeping condition $\Rightarrow$ restrict all suitable l to an “interval” of parallel lines in $O(n)$ time.



Reconstructing the input

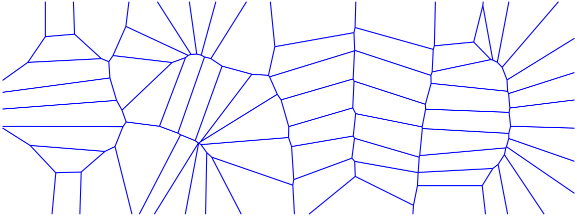
Theorem

GMP-SS can be solved and the set of solutions can be found in $O(n \log n)$ time of a $PSLG^\infty$ G with n edges.

Voronoi diagrams

Problem (GMP-VD)

Given a PSLG$^\infty$ G, can we find a set S of points such that $\mathcal{V}(S) = G$?



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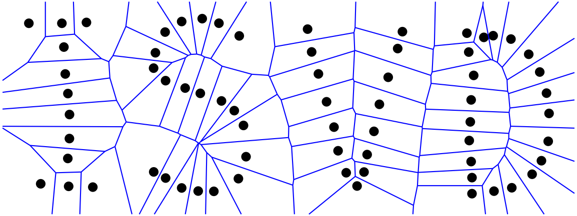

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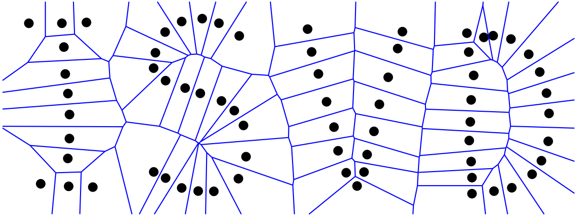

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- [Ash and Bolker, 1985]: Solve GMP-VD if all vertices have odd degree.

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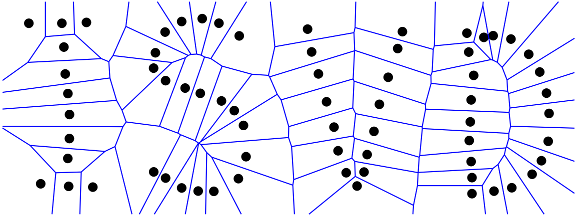


Prior work:

- [Ash and Bolker, 1985]: Solve GMP-VD if all vertices have odd degree.
- [Hartvigsen, 1992]: Solve GMP-VD by means of linear programming.

Characterization of Voronoi diagrams

We denote a solution of GMP-VD as a mapping $\rho\colon F \to \mathbb{R}^2$.

- We look for ρ such that $\mathcal{V}(\{\rho(f)\colon f \in F\}) = G$.

Lemma ([Ash and Bolker, 1985])

ρ solves GMP-VD if

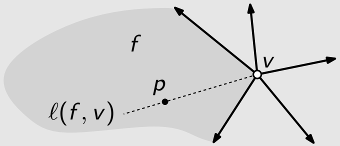
- **Inside-condition:** *$\rho(f) \in \operatorname{int} f$ for all $f \in F$.*
- **Bisector-condition:** *e is on the bisector of $\rho(f), \rho(f')$ for any edge $e = f \cap f'$.*

Recognizing Voronoi diagrams

Lemma

$\Phi_{b_1 \circ \cdots \circ b_d}(p) = p$ *if and only if*

$$\begin{cases} v \text{ fulfills the balance-condition} & \text{if } d \text{ is even} \\ p \in \ell(f, v) \text{ for some } f \in F \text{ with } v \in f & \text{if } d \text{ is odd} \end{cases} \quad (7)$$



Recognizing Voronoi diagrams

Lemma

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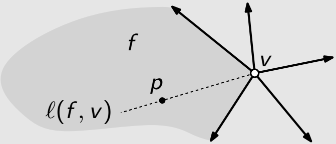

We again define

$$S(f) := (\operatorname{int} f) \cap \bigcap_{\substack{v \text{ is vertex of } f \\ \deg(v) \text{ is odd}}} \ell(f, v) \tag{8}$$

Easily becomes a single point.

$$X := \bigcap_{f \in F} \Phi_{f \rightsquigarrow T_r}(S(f)) \tag{9}$$

Every point implies a solution ρ.

Conclusion

Characterization of straight skeletons:

- Deeper insight in the geometry and structure of $\mathcal{S}(H)$.
- Allows for necessary and sufficient $O(n)$ time a-posteriori checks of the validity of $\mathcal{S}(H)$ in straight-skeleton codes.

We solve GMP-SS and GMP-VD on G

- using a unified framework based on reflections on edges of a spanning tree of the dual of G
- in $O(n \log n)$ time.
- First result on GMP-SS.
- Closes a gap in [Ash and Bolker, 1985] for GMP-VD when vertices have even degree.

Bibliography I

Aichholzer, O., Alberts, D., Aurenhammer, F., and Gärtner, B. (1995).
A novel type of skeleton for polygons.
J. Universal Comp. Sci., 1(12):752–761.

Aichholzer, O. and Aurenhammer, F. (1998).
Straight skeletons for general polygonal figures in the plane.
In Samoilenko, A., editor, *Voronoi's Impact on Modern Science, Book 2*, pages 7–21. Institute of Mathematics of the National Academy of Sciences of Ukraine, Kiev, Ukraine.

Aichholzer, O., Cheng, H., Devadoss, S. L., Hackl, T., Huber, S., Li, B., and Risteski, A. (2012).
What makes a tree a straight skeleton?
In *Proc. 24th Canad. Conf. on Comp. Geom. (CCCG '12)*, pages 267–272, Charlottetown, Canada.

Ash, P. and Bolker, E. (1985).
Recognizing Dirichlet Tesselations.
Geometriae Dedicata, 19:175–206.

Bibliography II

Edelsbrunner, H., Guibas, L., and Sharir, M. (1989).
The Upper Envelope of Piecewise Linear Functions: Algorithms and Applications.
Discrete Comput. Geom., 4(1):311–336.

Hartvigsen, D. (1992).
Recognizing Voronoi Diagrams with Linear Programming.
ORSA J. Computing, 4(4):369–374.

Hershberger, J. (1989).
Finding the Upper Envelope of n Line Segments in $O(n \log n)$ Time.
Inform. Process. Lett., 33(4):169–174.

Huber, S. (2012).
Computing Straight Skeletons and Motorcycle Graphs: Theory and Practice.
Shaker Verlag.
ISBN 978-3-8440-0938-5.

Characterization: Proof

Key idea: G and $\mathcal{S}(H)$ each impose a wavefront-propagation process, $\mathcal{W}_G(t)$ and $\mathcal{W}_{\mathcal{S}(H)}(t)$.

- Claim: they are identical

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Lemma

The initial wavefronts $\mathcal{W}_G(\epsilon)$ and $\mathcal{W}_{\mathcal{S}(H)}(\epsilon)$ are identical.

Characterization: Proof

Lemma

Assume that $\mathcal{W}_G(t') = \mathcal{W}_{\mathcal{S}(H)}(t')$ *for* $0 < t' < t$.

- *If* $\mathcal{W}_G(t)$ *hits a vertex* v *of* G^*, *then* v *coincides with a vertex of* $\mathcal{S}(H)$.
- *Analogously for* $\mathcal{W}_{\mathcal{S}(H)}$.

Characterization: Proof

Lemma

Assume that $\mathcal{W}_G(t') = \mathcal{W}_{\mathcal{S}(H)}(t')$ for $0 < t' < t$.

- *If $\mathcal{W}_G(t)$ hits a vertex v of G^*, then v coincides with a vertex of $\mathcal{S}(H)$.*
- *Analogously for $\mathcal{W}_{\mathcal{S}(H)}$.*

Theorem

$\mathcal{W}_G(t) = \mathcal{W}_{\mathcal{S}(H)}(t)$ for all t.

Proof. [Sketch]

- By induction on the chronological order when $\mathcal{W}_G$ resp. $\mathcal{W}_{\mathcal{S}(H)}$ hits a vertex v of G resp. $\mathcal{S}(H)$.
- In a neighborhood of v we have swept and not-yet-swept cones.
- Insight: In the not-yet-swept cones contain each exactly one "outgoing" edge of G resp. $\mathcal{S}(H)$.
- Claim: these edges are identical in the neighborhood of v.

Non-unique solutions to GMP-SS and GMP-VD