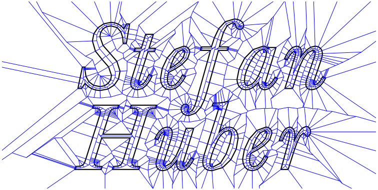
Computing Straight Skeletons and Motorcycle Graphs: Theory and Practice



Universität Salzburg, Austria

August 4, 2011

Motivation

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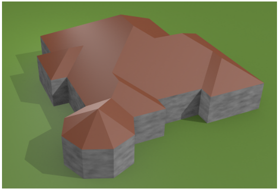

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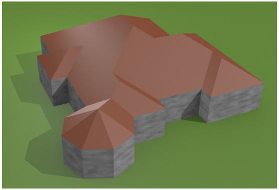

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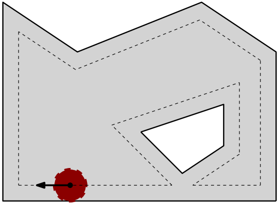

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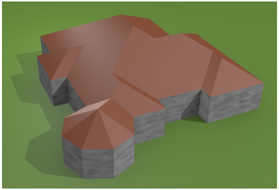

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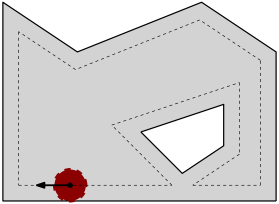

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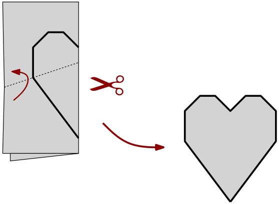

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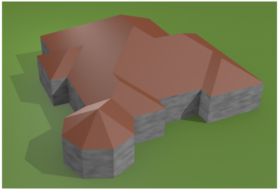

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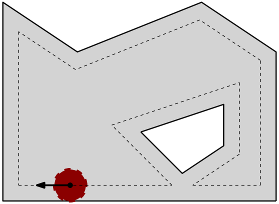

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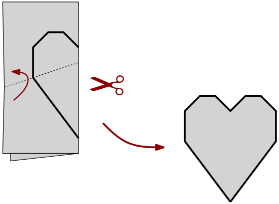


Fold-and-cut problem

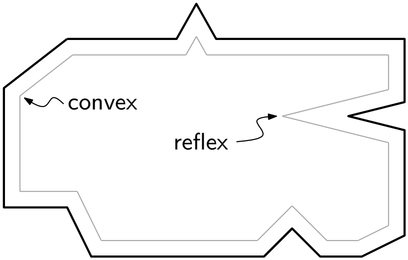
Using the straight skeleton, we obtain

- simple,
- efficient,
- numerically stable

algorithms for many problems in computational geometry.

What is the straight skeleton?

- Introduced by [Aichholzer et al., 1995] for simple polygons P.
- Definition based on **wavefront propagation process**:



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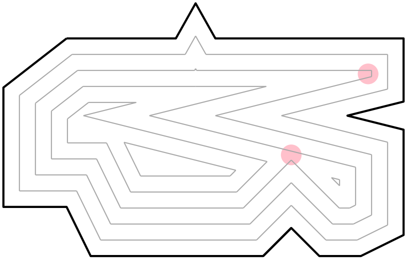

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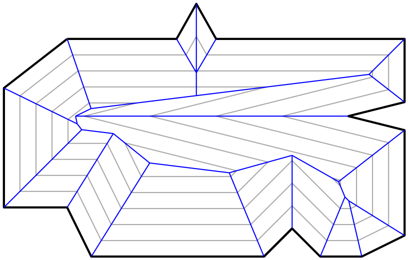

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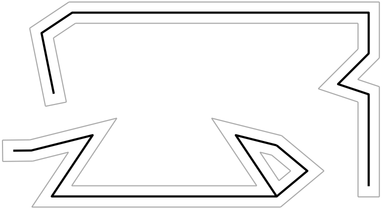
- Extended to PSLGs[1] G by [Aichholzer and Aurenhammer, 1996].
  - Defined on the entire plane.
  - Rectangular caps at terminal vertices.



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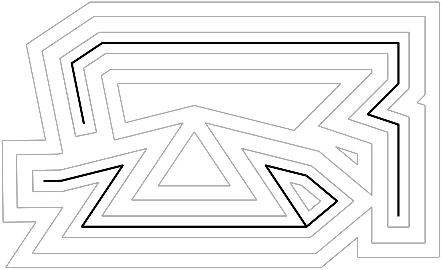

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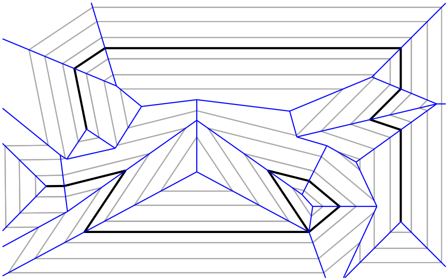

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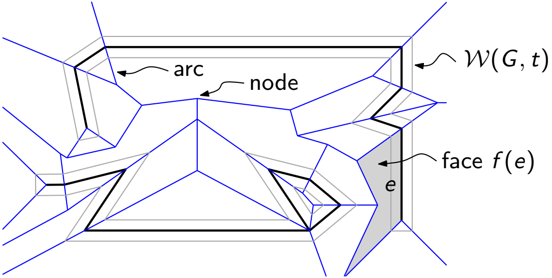

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- A **motorcycle** is a moving point with
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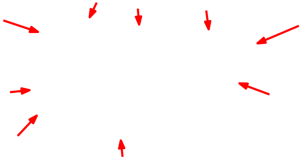

Motorcycle graph

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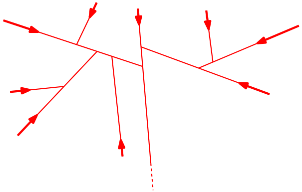
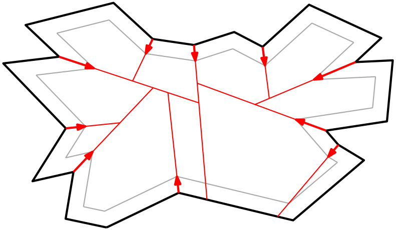
- A **motorcycle** is a moving point with
  - a start point and
  - a constant velocity.
- Consider n motorcycles $m_1, \ldots, m_n$.
  - Each motorcycle leaves a **trace** behind.
  - A motorcycle **crashes** when reaching another motorcycle's trace.
  - The **motorcycle graph** $\mathcal{M}(m_1, \ldots, m_n)$ is the arrangement of all traces.



Motorcycle graph

- Motorcycle graph $\mathcal{M}(P)$ induced by a polygon P:
  - A motorcycle for each reflex wavefront vertex, with the same velocity.
  - The edges of P are considered to be walls.



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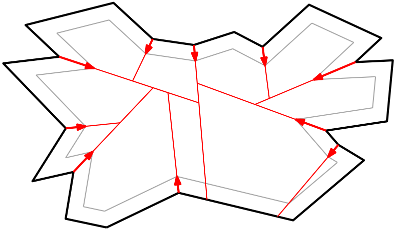

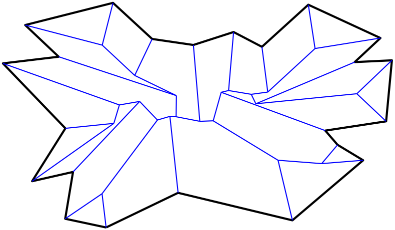

- P is **non-degenerate** if two or more motorcycles do not crash simultaneously into each other.

Theorem ([Cheng and Vigneron, 2002])

$\mathcal{M}(P)$ covers the reflex arcs of $\mathcal{S}(P)$.

Current situation

- **Known algorithms:**
  - $r \in O(n)$: number of reflex wavefront vertices.
  - Impl.: algorithm is suitable for implementation.

Alg.	Time	PSLG	Impl.
[Aichholzer et al., 1995]	$O(nr \log n)$	No	Yes
[Aichholzer and Aurenhammer, 1996]	$O(n^3 \log n)$	Yes	Yes
[Eppstein and Erickson, 1999]	$O(n^{1+\epsilon} + n^{8/11+\epsilon} r^{9/11+\epsilon})$	Yes	No
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- **Known implementations:**
  CGAL: for polygons with holes, quadratic runtime and memory usage.

Road map

Objective: Find an algorithm which is

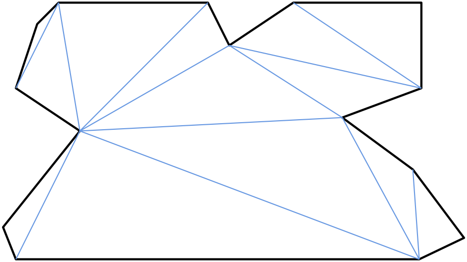
- fast in practice and
- simple enough to be implemented.

Approach:

- Algorithm by Aichholzer and Aurenhammer sounds promising.
- However, worst-case time complexity: $O(n^3 \log n)$.
- Open question: is the bound tight?

Triangulation-based approach

- Presented by [Aichholzer and Aurenhammer, 1996], handles PSLGs as input.



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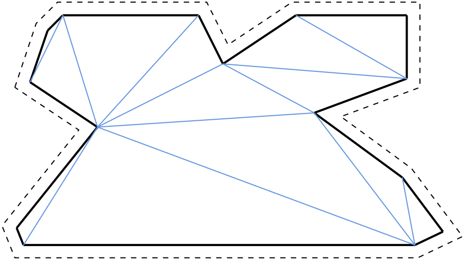

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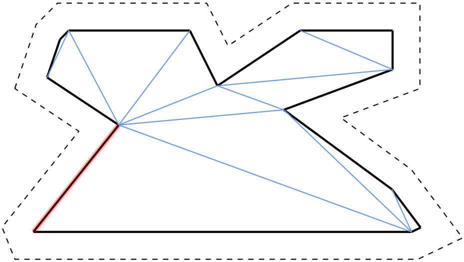

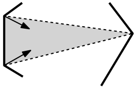

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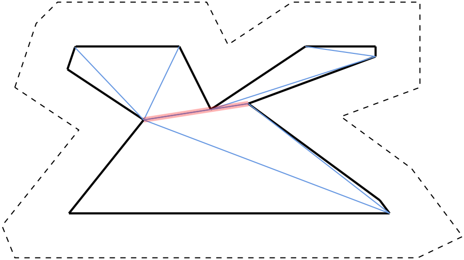

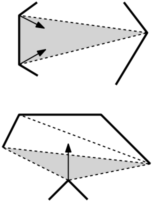

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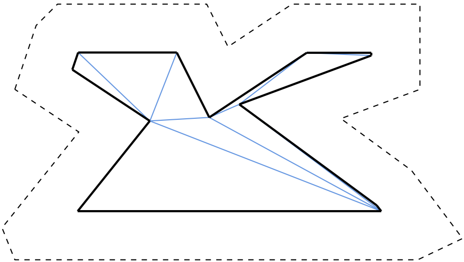

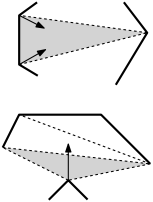

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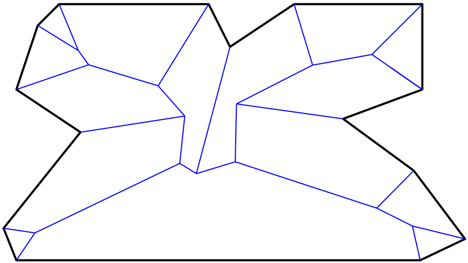

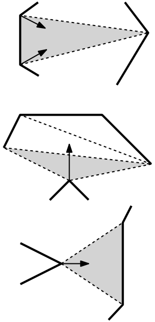

Triangulation-based approach

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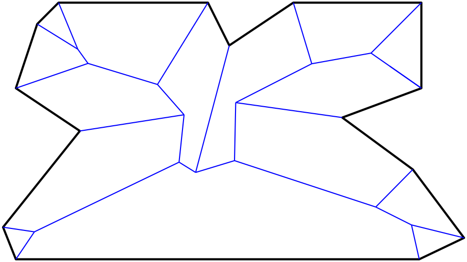
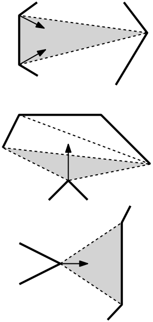
- Kinetic triangulation:
  - Edge events: $O(n)$
  - Flip events: $O(n^3)$
  - Split events: $O(n)$
  - Results in an $O(n^3 \log n)$ runtime.





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Lemma ([Huber and Held, 2010b])

There exists polygons P with n vertices and triangulations T of P such that $\Omega(n)$ diagonals each reappear $\Omega(n)$ times.

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Lemma ([Huber and Held, 2010b])

Every PSLG G with n vertices admits a triangulation that employs $O(n)$ Steiner points and is free of flip events.

Proof idea:

- Add the nodes and arcs of $\mathcal{S}(G)$ as Steiner points and triangulation diagonals.
- Show that the faces of $\mathcal{S}(G)$ can be triangulated such that flip events are avoided.
- Why does that work? Reflex wavefront vertices move on diagonals!

A novel wavefront-type algorithm

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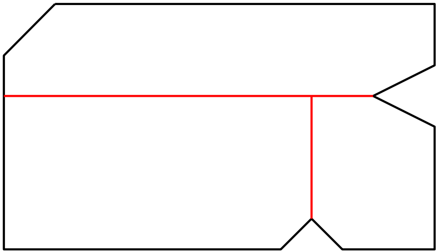

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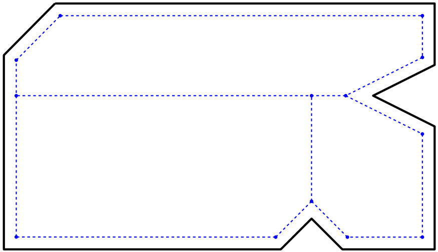
- Simulate the **extended wavefront**. [Huber and Held, 2010a]



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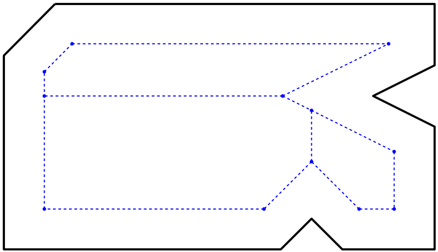

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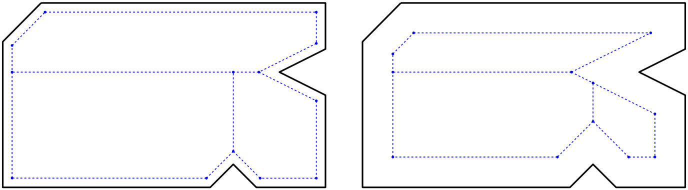

Lemma ([Huber and Held, 2010a])

The faces of the extended wavefront are convex at any time.

- Hence, in the simulation, we only need to consider neighboring vertices.

Theorem ([Cheng and Vigneron, 2002])

$\mathcal{M}(P)$ *covers the reflex arcs of* $\mathcal{S}(P)$.

- Hence, split events happen within the motorcycle traces.

A novel wavefront-type algorithm

Works only for a restricted class of input data:

- No arbitrary polygons.
- No PSLGs.

Idea

Generalize $\mathcal{M}(P)$ of non-degenerate polygons P to $\mathcal{M}(G)$ of arbitrary PSLGs G!

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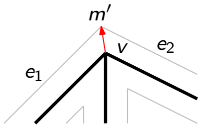
Idea

Generalize $\mathcal{M}(P)$ of non-degenerate polygons P to $\mathcal{M}(G)$ of arbitrary PSLGs G!

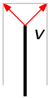
However:

- $\mathcal{M}(G)$ needs to cover all reflex arcs of $\mathcal{S}(G)$!
- The extended wavefront still needs to induce a convex tessellation!

- (a–b) Place a motorcycle for each reflex wavefront vertex in $\mathcal{W}(G, 0)$.

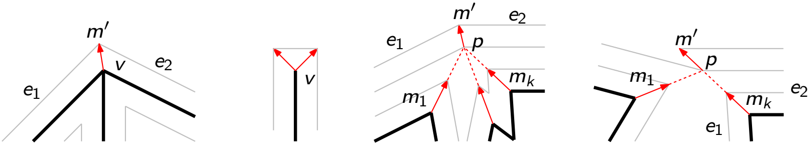


(a)



(b)

- (a–b) Place a motorcycle for each reflex wavefront vertex in $\mathcal{W}(G, 0)$.
- Launch a new motorcycle when multiple motorcycles meet:
  - (c) To cover reflex straight skeleton arcs.
  - (d) To respect the convex tessellation property.



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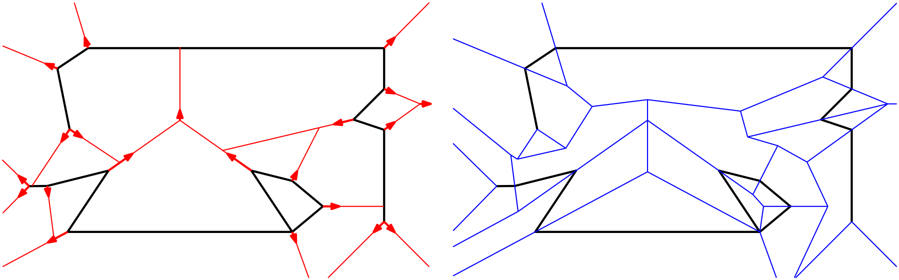

Lemma ([Huber and Held, 2011c])

Consider a point p in the relative interior of $\mathcal{M}(G)$. Then a local disc is tessellated into convex slices by $\mathcal{M}(G)$.

Theorem ([Huber and Held, 2011c])

$\mathcal{M}(G)$ covers the reflex arcs of $\mathcal{S}(G)$.

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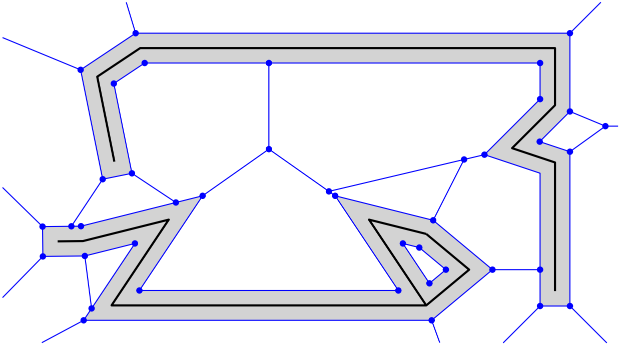
- **Computing $\mathcal{S}(G)$ using graphics hardware:**
  - Apply techniques used by [Hoff et al., 1999] for Voronoi diagrams.

Applications of the theorem

- **A non-procedural alternative characterization of $\mathcal{S}(G)$:**
  - Generalizes a result of [Cheng and Vigneron, 2002].
- **Computing $\mathcal{S}(G)$ using graphics hardware:**
  - Apply techniques used by [Hoff et al., 1999] for Voronoi diagrams.
- **A novel wavefront-type algorithm to compute $\mathcal{S}(G)$.**

A wavefront-type algorithm

We simulate the propagation of the extended wavefront:

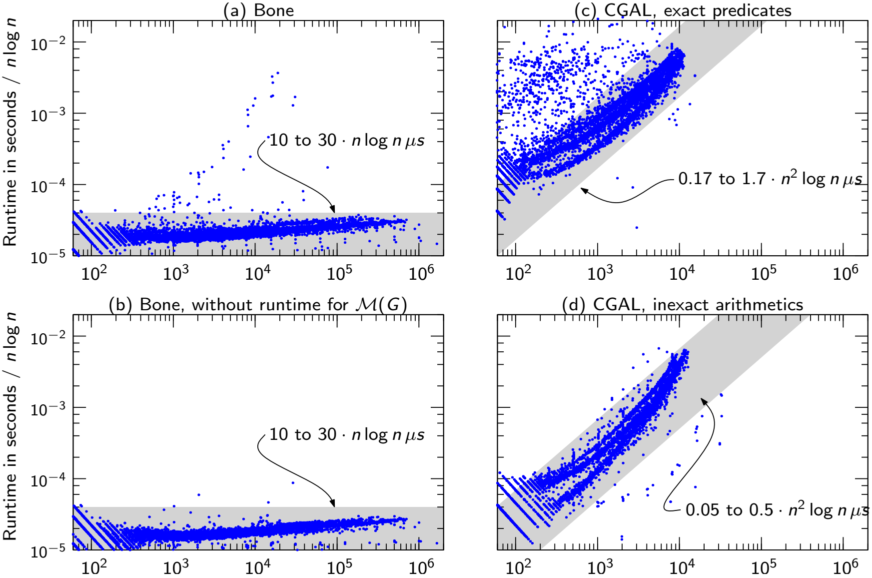


- Split events happen within the extended wavefront.
- Only neighboring vertices can meet.

BONE: a C++ implementation:

- Double-precision floating-point arithmetics.
- $O(n)$ space complexity.
- $O(n \log n)$ runtime in practice.
- $O(nr \log n)$ runtime in the worst case.
  - r is the number of reflex wavefront vertices.
  - Unlikely to occur in the real-world.

Runtime


(a) Bone
Runtime in seconds / $n \log n$
10^{-2}
10^{-3}
10^{-4}
10^{-5}
$10 \text{ to } 30 \cdot n \log n\ \mu s$
10^2 10^3 10^4 10^5 10^6
(c) CGAL, exact predicates
$0.17 \text{ to } 1.7 \cdot n^2 \log n\ \mu s$
(b) Bone, without runtime for $\mathcal{M}(G)$
$10 \text{ to } 30 \cdot n \log n\ \mu s$
(d) CGAL, inexact arithmetics
$0.05 \text{ to } 0.5 \cdot n^2 \log n\ \mu s$

Memory consumption

Random polygons generated by RPG.

Size n	**Bone** MB	**Bone** factor	**CGAL** MB	**CGAL** factor
256	1.44		3.77	
512	2.65	1.8x	13.4	3.5x
1 024	5.06	1.9x	51.1	3.8x
2 048	9.86	1.9x	201	3.9x
4 096	19.5	2.0x	792	3.9x
8 192	38.7	2.0x	3 197	4.0x
16 384	77.1	2.0x	12 600	3.9x

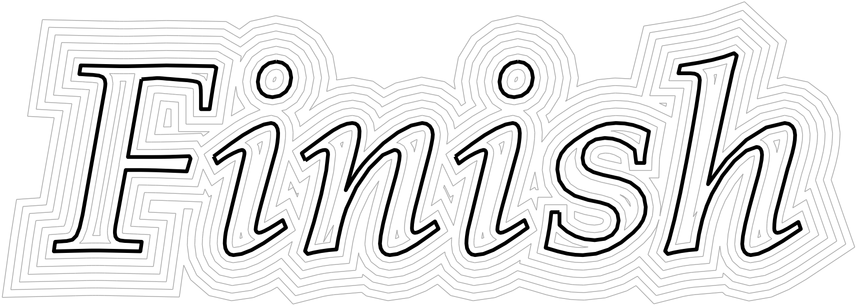
Table: Memory usage of Bone and CGAL

Summary

- Investigations of the triangulation-based algorithm:
  - Huber, S. and Held, M. (2010b). Straight skeletons and their relation to triangulations. In *Proc. 26th Europ. Workshop on Comp. Geom. (EuroCG '10)*, pages 189–192, Dortmund, Germany.
- Generalization of the motorcycle graph concept:
  - Alternative characterization of $\mathcal{S}(G)$.
  - Motivates an algorithm based on graphics hardware.
  - Huber, S. and Held, M. (2011c). Theoretical and practical results on straight skeletons of planar straight-line graphs. In *Proc. 27th Annual Symp. on Comp. Geom. (SoCG '11)*, pages 171–178, Paris, France.
- A novel wavefront-type algorithm and the implementation Bone:
  - Accepts arbitrary PSLGs as input.
  - The fastest and most memory-efficient implementation at the moment.
  - Huber, S. and Held, M. (2010a). Computing straight skeletons of planar straight-line graphs based on motorcycle graphs. In *Proc. 22nd Canad. Conf. on Comp. Geom. (CCCG '10)*, pages 187–190, Winnipeg, Canada.
  - Huber, S. and Held, M. (2011c). Theoretical and practical results on straight skeletons of planar straight-line graphs. In *Proc. 27th Annual Symp. on Comp. Geom. (SoCG '11)*, pages 171–178, Paris, France.

Summary: topics not covered in the talk

- MOCA: a motorcycle graph implementation:
  - Exploits geometric hashing, which is motivated by a stochastic analysis of the mean trace length.
  - MOCA: currently the only implementation which exhibits a subquadratic runtime in practice.
  - Huber, S. and Held, M. (2009). A practice-minded approach to computing motorcycle graphs. In *Proc. 25th Europ. Workshop on Comp. Geom. (EuroCG '09)*, pages 305–308, Brussels, Belgium.
  - Huber, S. and Held, M. (2011b). Motorcycle graphs: Stochastic properties motivate an efficient yet simple implementation. *ACM Journal on Exp. Alg.*, 17. in press.
- Straight skeletons approximating motorcycle graphs:
  - We are given n motorcycles. We can determine a G such that $\mathcal{S}(G)$ approximates $\mathcal{M}(m_1, \ldots, m_n)$.
  - We have an algorithm that computes $\mathcal{M}(m_1, \ldots, m_n)$ using $\mathcal{S}(G)$.
  - We obtained a proof for the P-completeness of straight skeletons of polygons with holes.
  - Huber, S. and Held, M. (2011a). Approximating a motorcycle graph by a straight skeleton. In *Proc. 23rd Canad. Conf. on Comp. Geom. (CCCG '11)*, Toronto, Canada. to appear.


Finish

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