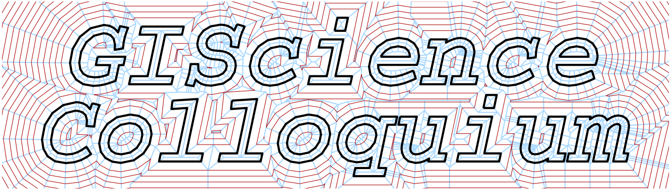
Skeleton Structures in Computational Geometry

An introduction with GIS in mind

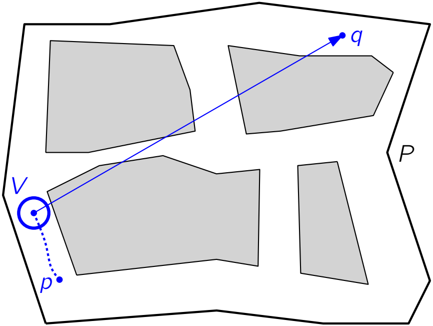
Stefan Huber

January 7, 2013



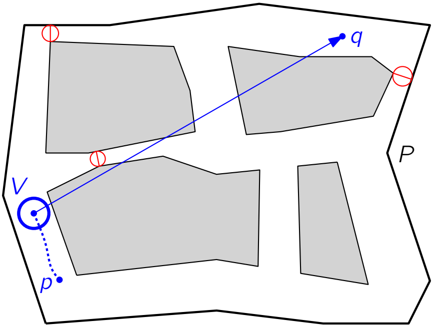
Motivation

Is there a path for the vehicle V from p to q within the polygon with holes, P?



Motivation

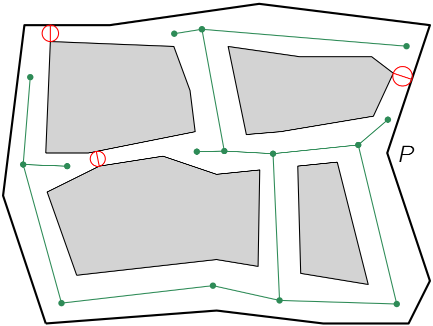
Is there a path for the vehicle V from p to q within the polygon with holes, P?



- We would like to have a geometric structure that
  - lets us easily identify "bottlenecks" of P and

Motivation

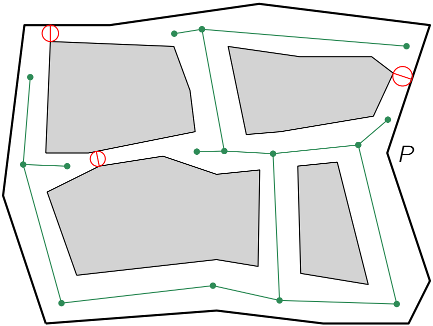
Is there a path for the vehicle V from p to q within the polygon with holes, P?



- We would like to have a geometric structure that
  - lets us easily identify "bottlenecks" of P and
  - allows us to reuse known path-finding algorithms (on graphs).

Motivation

Is there a path for the vehicle V from p to q within the polygon with holes, P?

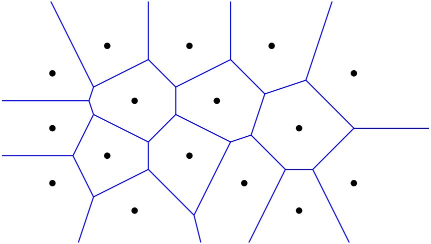


- We would like to have a geometric structure that
  - lets us easily identify "bottlenecks" of P and
  - allows us to reuse known path-finding algorithms (on graphs).
- For short: we need information about the **shape** of P.
  - Skeleton structures can do that.

Generalized Voronoi diagrams

The Voronoi diagram of points in the Euclidean plane has been generalized in multiple ways:

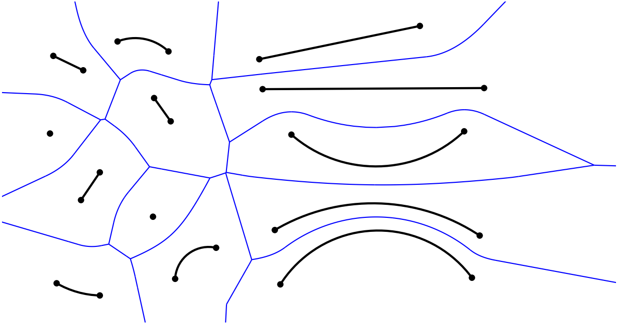
- plane $\rightarrow$ higher dimensions
- points $\rightarrow$ straight-line segments, circular arcs, ...
- Euclidean $\rightarrow$ L^k-norms, convex distance functions, ...



In this talk: Voronoi diagram of points, straight-line segments and circular arcs in the Euclidean plane.

Generalized Voronoi diagrams

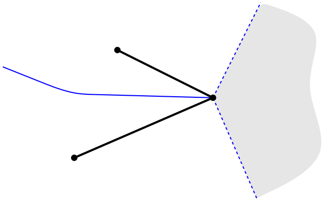
Given: set S of **input sites**, i.e., points, straight-line segments, circular arcs, not intersecting in their relative interior.



- Plane tessellated into **cells around input sites**.
  - Points within the cell of site i are **closer to** i than to all other sites.
  - Bisectors are parabolic/elliptic arcs.
- $\mathcal{V}(S)$ is the Voronoi diagram of S. That is, $\mathcal{V}(S)$ consists of the boundaries of all cells.

Generalized Voronoi diagrams: details

What if sites touch?

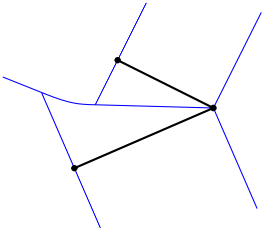


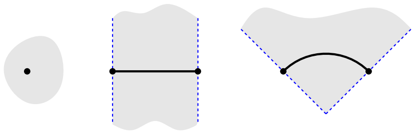
Any point in the shaded area is equidistant to both segments.

Generalized Voronoi diagrams: details

What if sites touch?

- Endpoints of segments and arcs are input sites, too.
- Voronoi cell of a site is restricted to "cone of influence".

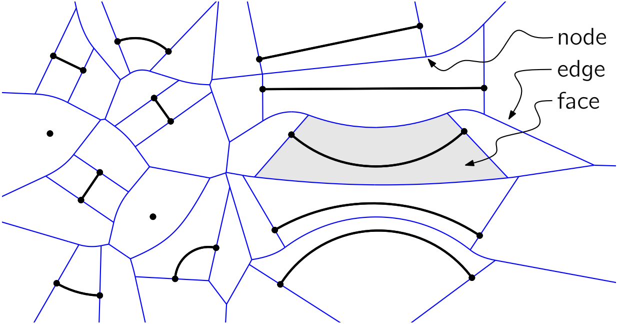




Generalized Voronoi diagrams: details

What if sites touch?

- Endpoints of segments and arcs are input sites, too.
- Voronoi cell of a site is restricted to “cone of influence”.

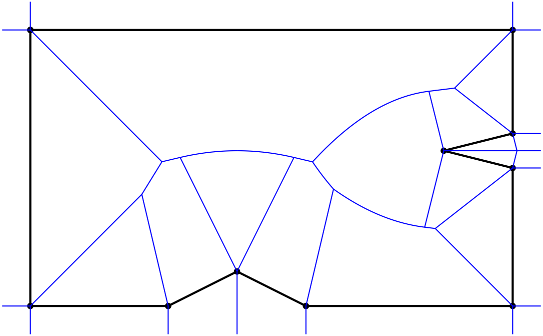


Precise definition in [Held and Huber, 2009].

Voronoi diagram of a shape

We are given a simple polygon P.

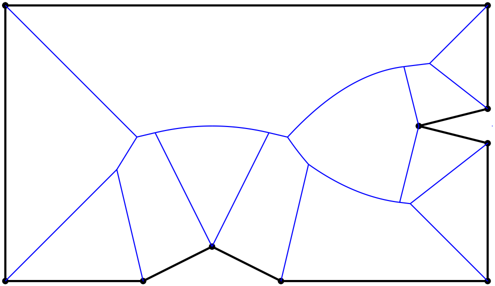
- P consists of vertices and edges $\rightarrow$ take them as the set S of input sites.
- Short-hand notation: $\mathcal{V}(P)$ is the resulting Voronoi diagram, i.e., $\mathcal{V}(S)$.
  - P is tessellated into Voronoi cells.



Voronoi diagram of a shape

We are given a simple polygon P.

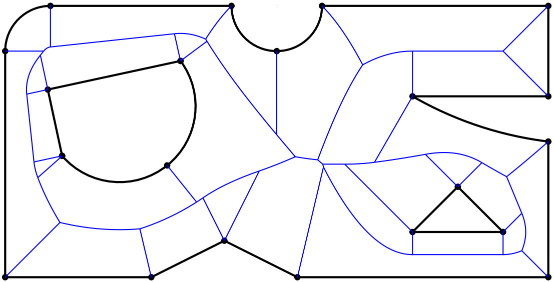
- P consists of vertices and edges $\rightarrow$ take them as the set S of input sites.
- Short-hand notation: $\mathcal{V}(P)$ is the resulting Voronoi diagram, i.e., $\mathcal{V}(S)$.
  - P is tessellated into Voronoi cells.
  - Sometimes $\mathcal{V}(P)$ is considered to be restricted to P.



Voronoi diagram of a shape

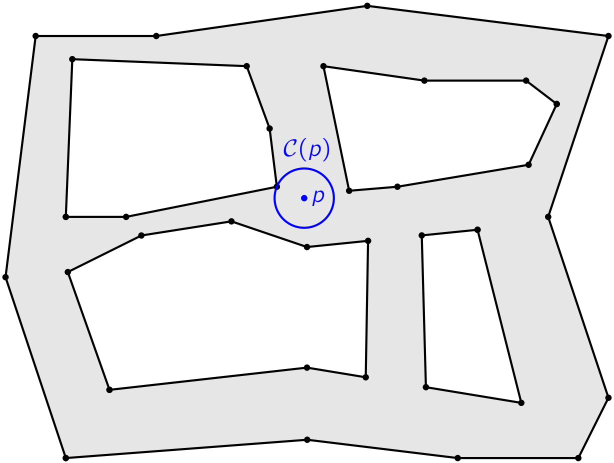
We are given a simple polygon P.

- P consists of vertices and edges $\rightarrow$ take them as the set S of input sites.
- Short-hand notation: $\mathcal{V}(P)$ is the resulting Voronoi diagram, i.e., $\mathcal{V}(S)$.
  - P is tessellated into Voronoi cells.
  - Sometimes $\mathcal{V}(P)$ is considered to be restricted to P.
  - Sometimes, P may have holes and its boundary may also comprise circular arcs.



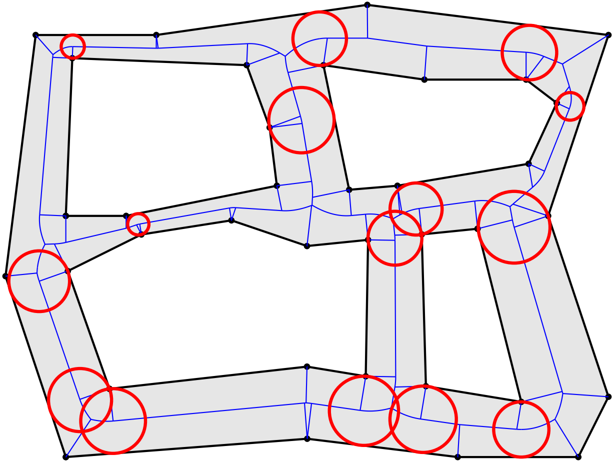
Finding bottlenecks of shapes

- The **clearance disk** $\mathcal{C}(p)$ is the largest disk within P centered at the point p.
  - Its radius is the **clearance radius**.



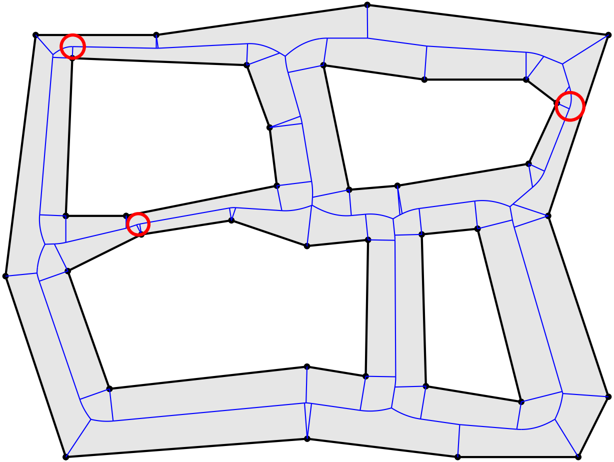
Finding bottlenecks of shapes

- The **clearance disk** $\mathcal{C}(p)$ is the largest disk within P centered at the point p.
  - Its radius is the **clearance radius**.
- **Bottleneck**: a Voronoi node with locally minimal clearance radius.



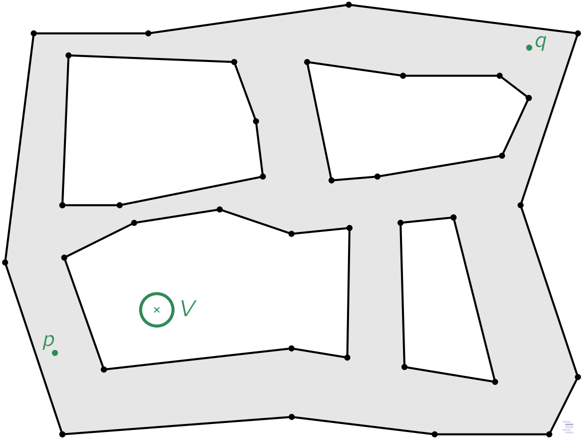
Finding bottlenecks of shapes

- The **clearance disk** $\mathcal{C}(p)$ is the largest disk within P centered at the point p.
  - Its radius is the **clearance radius**.
- **Bottleneck**: a Voronoi node with locally minimal clearance radius.



Collision-free paths

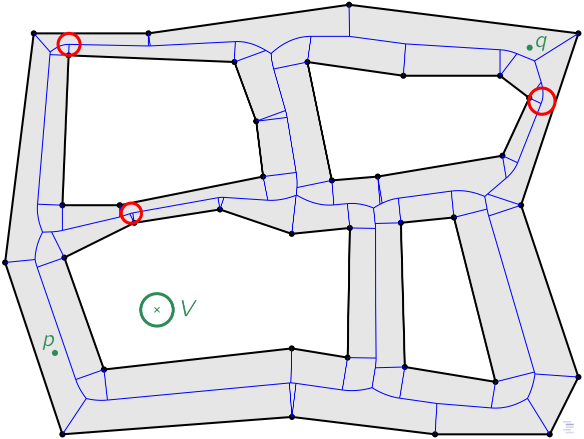
Can we move V form p to q within P?



Collision-free paths

Can we move V form p to q within P?

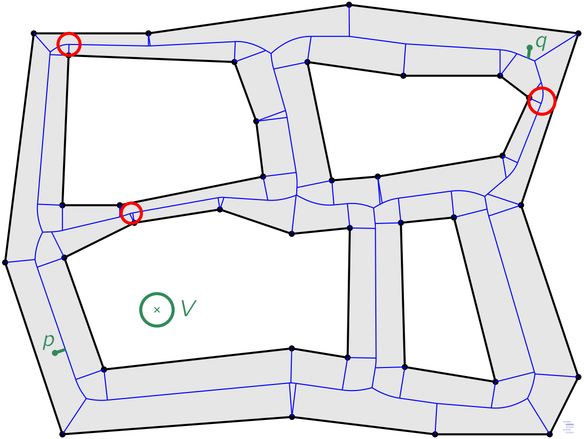
- Compute $\mathcal{V}(P)$. Consider the bottlenecks, whose clearance disk is smaller than V, removed.



Collision-free paths

Can we move V form p to q within P?

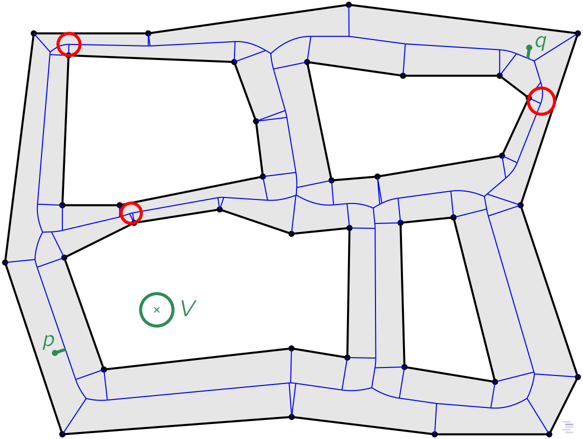
- Compute $\mathcal{V}(P)$. Consider the bottlenecks, whose clearance disk is smaller than V, removed.
- Project p resp. q to points p' resp. q' on Voronoi edges.



Collision-free paths

Can we move V form p to q within P?

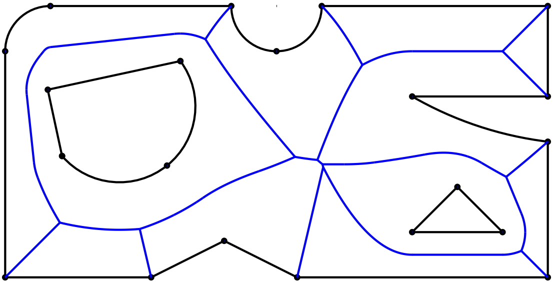
- Compute $\mathcal{V}(P)$. Consider the bottlenecks, whose clearance disk is smaller than V, removed.
- Project p resp. q to points p' resp. q' on Voronoi edges.
- Find a path from p' to q' on $\mathcal{V}(P)$ using ordinary graph algorithms.



Medial axis

We are given a shape P.

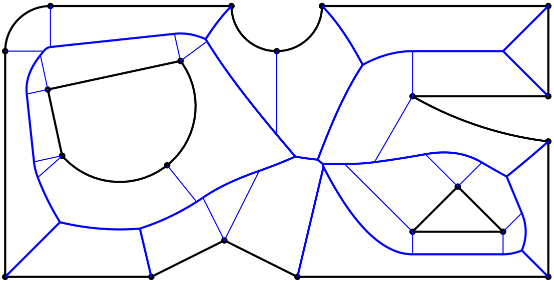
- The medial axis $\mathcal{M}(P)$ consists of those points p within P whose minimum distance to the boundary of P is assumed at two or more boundary points.
- That is, the clearance disk at p touches P at two or more points.
- Hence, $\mathcal{M}(P) \subseteq \mathcal{V}(P)$.
  - $\mathcal{M}(P)$ is easily extracted from $\mathcal{V}(P)$.



Medial axis

We are given a shape P.

- The medial axis $\mathcal{M}(P)$ consists of those points p within P whose minimum distance to the boundary of P is assumed at two or more boundary points.
- That is, the clearance disk at p touches P at two or more points.
- Hence, $\mathcal{M}(P) \subseteq \mathcal{V}(P)$.
  - $\mathcal{M}(P)$ is easily extracted from $\mathcal{V}(P)$.

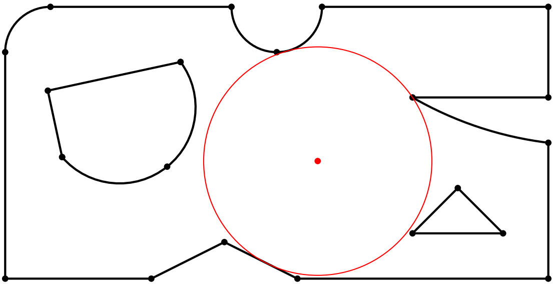


Maximum inscribed circle

What is the largest disk we can place in a shape P?

- Kind of complementary problem to bottleneck detection.

Algorithm:



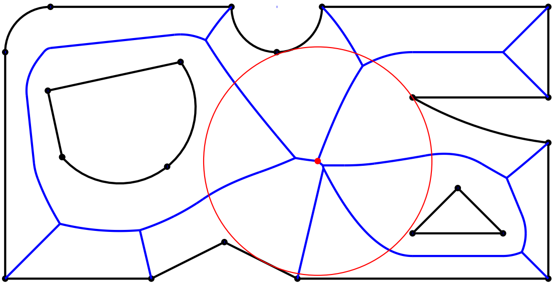
Maximum inscribed circle

What is the largest disk we can place in a shape P?

- Kind of complementary problem to bottleneck detection.

Algorithm:

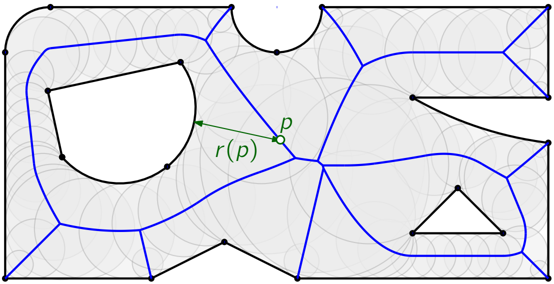
- Take the Voronoi node with largest clearance radius.



Reconstruction property, topological view

P equals the union of all clearance disks placed on $\mathcal{M}(P)$.

- $\mathcal{M}(P)$ and the clearance radius function $r(.)$ on $\mathcal{M}(P)$ can together reconstruct P.
- Besides "thickness", $\mathcal{M}(P)$ contains the essential information of the "shape" of P.
  - For instance, holes in P correspond to cycles in $\mathcal{M}(P)$.



Reconstruction property, topological view

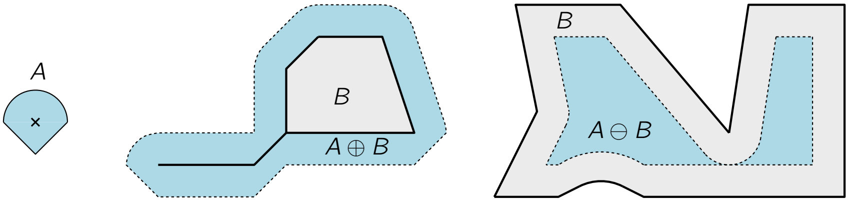
We are given networks of rivers by their polygonal shapes.

- How to find the center-lines of the rivers?
- How to find the main branches?
- How to prune small creeks?
- Which river is connected with which?
- How to remove details from a map, i.e., collapse small rivers or streets to a line?

Minkowski-sum and Minkowski-difference

The **Minkowski-sum** $A \oplus B$ of two sets A and B is the union of all B moved by a vector $v \in A$. Note that $B \oplus A = A \oplus B$.

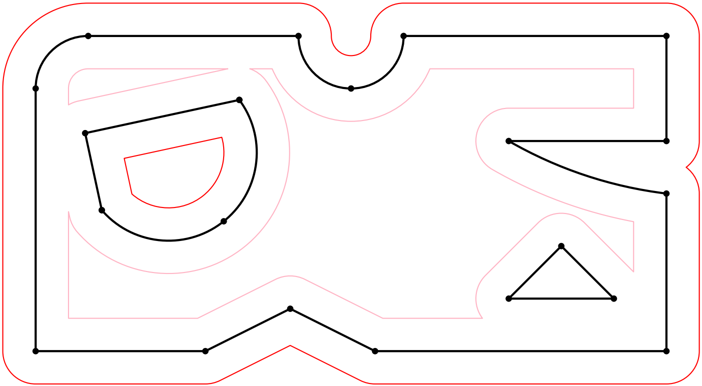
The **Minkowski-difference** $A \ominus B$ of two sets A and B is the largest set such that its Minkowski-sum with B is contained in A.



Offsetting

Let D_r denote the disk with radius r and the origin as center.

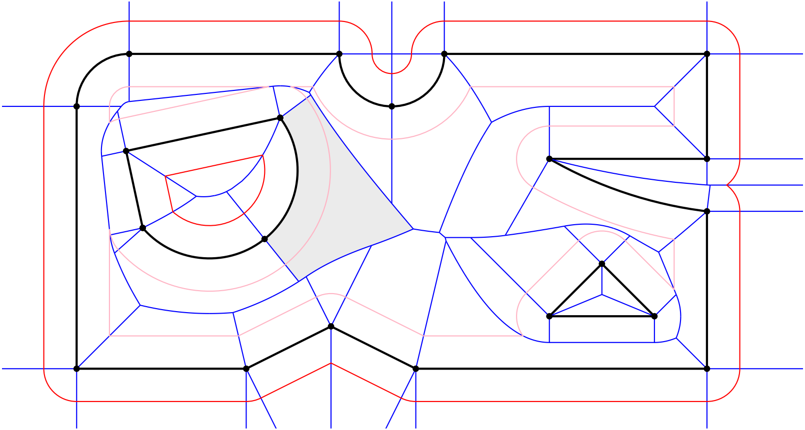
- How to compute all points inside/outside of P that have a distance of exactly (or at most) r? That is, **how to compute** $P \oplus D_r$ **resp.** $P \ominus D_r$**?**



Offsetting

Let D_r denote the disk with radius r and the origin as center.

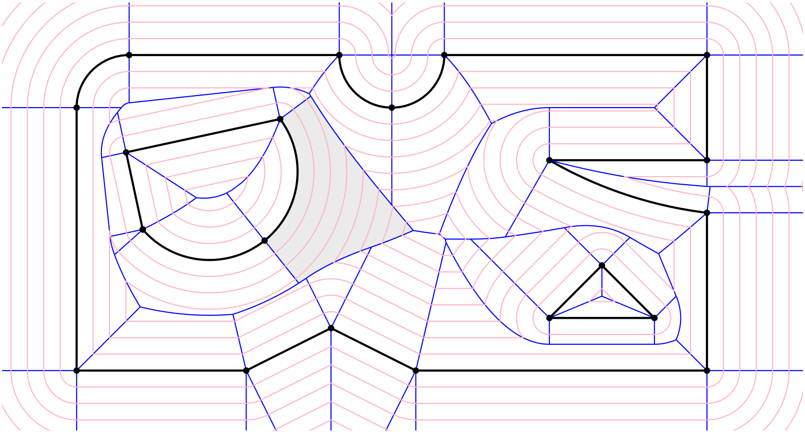
- How to compute all points inside/outside of P that have a distance of exactly (or at most) r? That is, **how to compute $P \oplus D_r$ resp. $P \ominus D_r$?**



Offsetting

Let D_r denote the disk with radius r and the origin as center.

- How to compute all points inside/outside of P that have a distance of exactly (or at most) r? That is, **how to compute** $P \oplus D_r$ **resp.** $P \ominus D_r$**?**



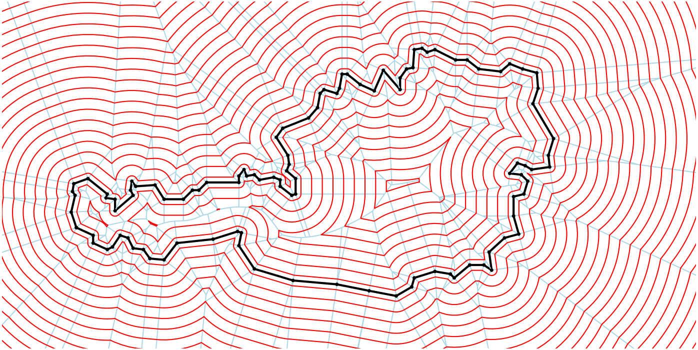
Offsetting

Collision-free paths: Note that there is a collision-free path for V from p to q if there is any path from p to q within $P \ominus V$.

Offsetting

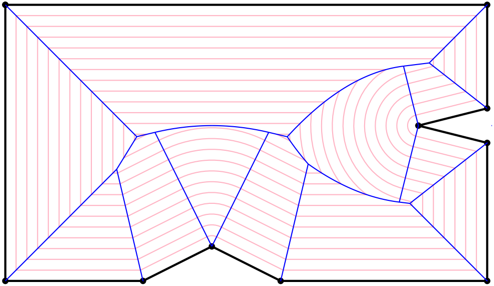
Collision-free paths: Note that there is a collision-free path for V from p to q if there is any path from p to q within $P \ominus V$.

Offsets have many **more applications**: computing tolerance zones, tool-paths in NC-machining, buffering in maps, thinning or extruding shapes, ...



Straight skeletons: Definition

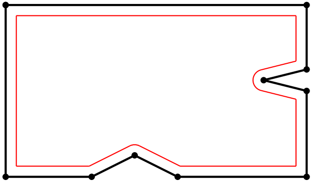
- The Voronoi diagram of a simple polygon P can be characterized by means of **interference patterns** of **offset segments** $\rightarrow$ **wavefront propagation**.



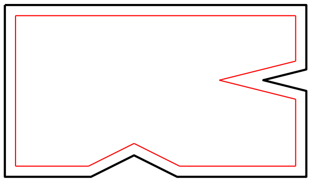
Straight skeletons: Definition

Suppose that we do not like circular arcs in the offset curves.

- Replace circular arcs by straight-line caps.
- Each **wavefront edge** is parallel to an edge of P and moves with equal speed.



(a) Voronoi diagram

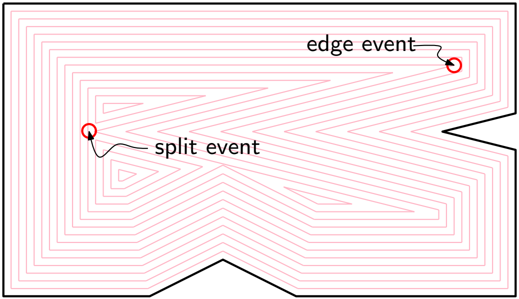


(b) straight skeleton

- The straight skeleton is defined by interference patterns of "mitered-offset curves".

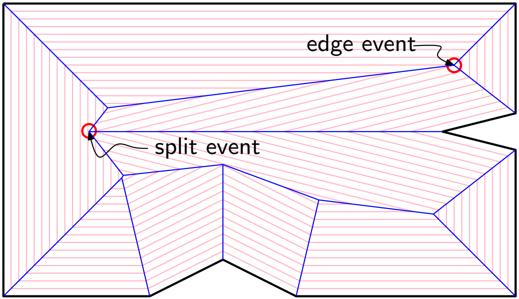
Straight skeletons: Definition

- Topological changes (events) during the wavefront propagation:
  - **Edge event:** a wavefront edge shrinks to zero length.
  - **Split event:** a reflex wavefront vertex splits another wavefront edge.



Straight skeletons: Definition

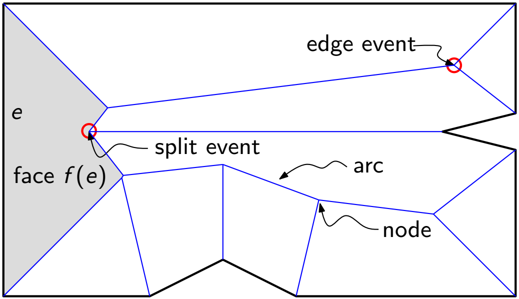
- Topological changes (events) during the wavefront propagation:
  - **Edge event:** a wavefront edge shrinks to zero length.
  - **Split event:** a reflex wavefront vertex splits another wavefront edge.



- Straight skeleton $\mathcal{S}(P)$: set of loci of all wavefront vertices.

Straight skeletons: Definition

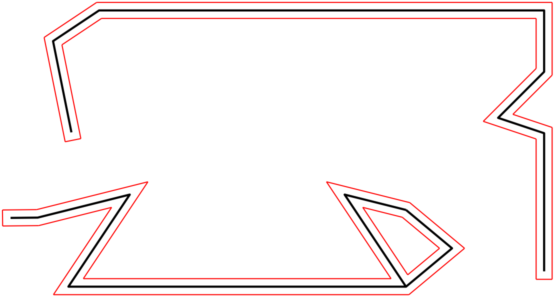
- Topological changes (events) during the wavefront propagation:
  - **Edge event:** a wavefront edge shrinks to zero length.
  - **Split event:** a reflex wavefront vertex splits another wavefront edge.



- Straight skeleton $\mathcal{S}(P)$: set of loci of all wavefront vertices.
  - Each **node** of $\mathcal{S}(P)$ is the locus of an event.
  - Each **arc** is on the bisector of polygon edges.
  - Each **face** is swept out by the wavefront edge emanated by an input edge.

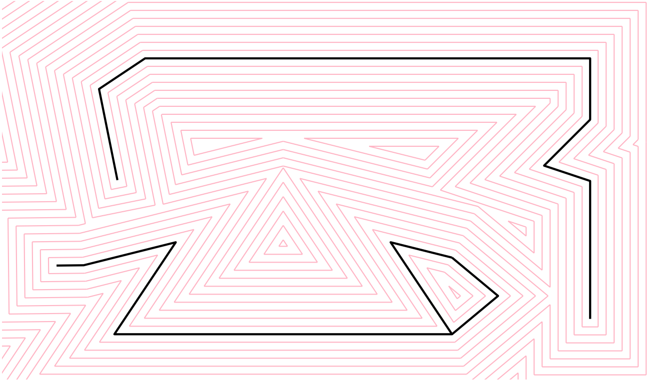
Straight skeleton of a PSLG

- PSLG: planar straight-line graph, i.e., a bunch of straight-line segments that do not intersect in their relative interior.
- [Aichholzer and Aurenhammer, 1998]: straight skeleton $\mathcal{S}(G)$ of a PSLG G
  - Each input edge sends out two parallel wavefront copies.
    - Each terminal vertex sends out an additional wavefront edge.



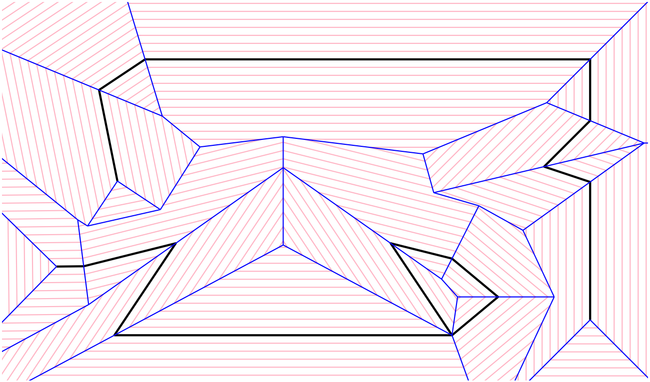
Straight skeleton of a PSLG

- PSLG: planar straight-line graph, i.e., a bunch of straight-line segments that do not intersect in their relative interior.
- [Aichholzer and Aurenhammer, 1998]: straight skeleton $\mathcal{S}(G)$ of a PSLG G
  - Each input edge sends out two parallel wavefront copies.
    - Each terminal vertex sends out an additional wavefront edge.



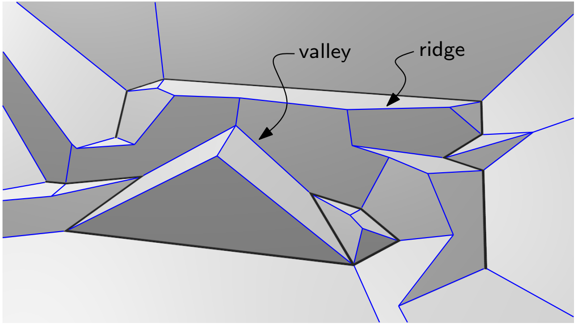
Straight skeleton of a PSLG

- PSLG: planar straight-line graph, i.e., a bunch of straight-line segments that do not intersect in their relative interior.
- [Aichholzer and Aurenhammer, 1998]: straight skeleton $\mathcal{S}(G)$ of a PSLG G
  - Each input edge sends out two parallel wavefront copies.
    - Each terminal vertex sends out an additional wavefront edge.



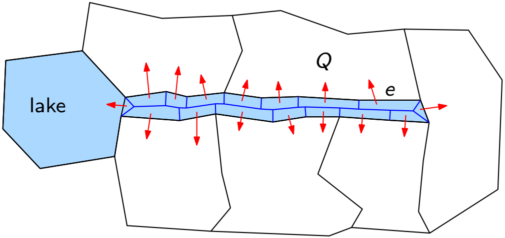
Straight skeletons: the terrain model

- Consider the wavefront propagation in three-dimensional space-time, with the z-axis representing time.
- An isoline of the resulting $\mathcal{T}(G)$ corresponds to the wavefront at some point in time.
- Projecting the valleys and ridges onto $\mathbb{R}^2$ gives us $\mathcal{S}(G)$ again.
  - Knowing $\mathcal{T}(G)$ is equivalent to knowing $\mathcal{S}(G)$.



Area collapsing

- Let us consider a map with a river given as a polygonal area.
- How to reduce the level of detail by collapsing the river's area to a line?
- [Haunert and Sester, 2008]:
  - Compute $\mathcal{S}(P)$[1], which tessellates P into faces.
  - To each edge e of P belongs a face $f(e)$ and each edge e also belongs to a neighboring polygon Q.
  - Add $f(e)$ to Q.

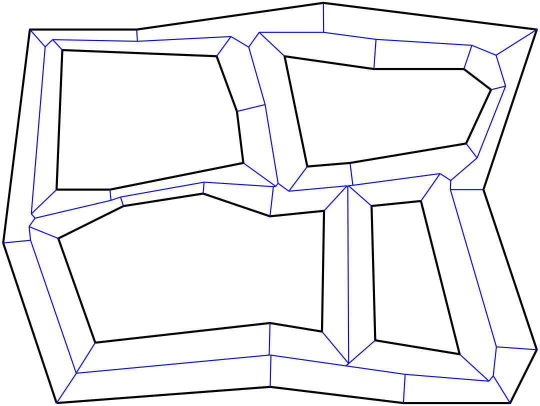


[1] Actually, weighted straight skeletons are employed.

Centerlines of roads

Similar problem to area collapsing:

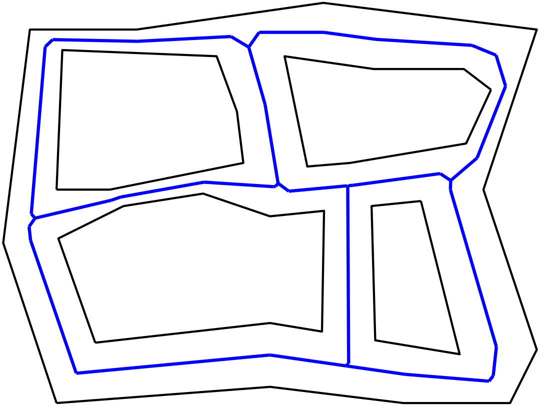
- Let us consider a map with a road network where roads are given by polygonal areas.
- [Haunert and Sester, 2008]: compute centerlines of roads resp. extract the corresponding network graph using straight skeletons.



Centerlines of roads

Similar problem to area collapsing:

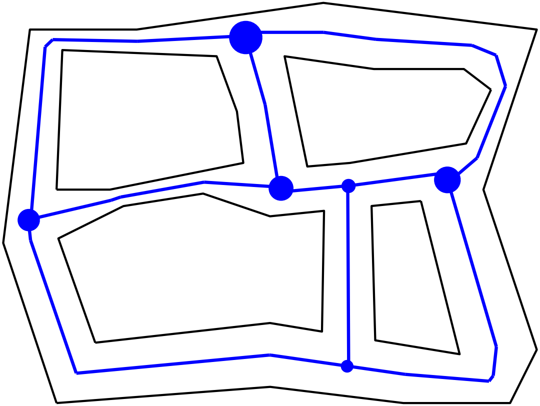
- Let us consider a map with a road network where roads are given by polygonal areas.
- [Haunert and Sester, 2008]: compute centerlines of roads resp. extract the corresponding network graph using straight skeletons.



Centerlines of roads

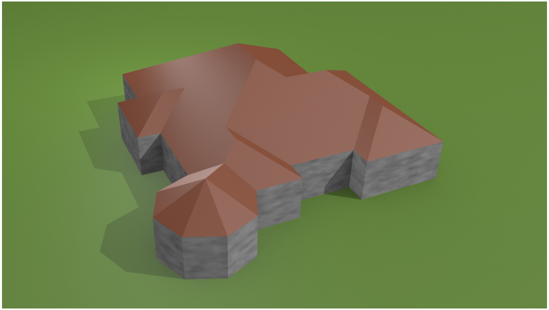
Similar problem to area collapsing:

- Let us consider a map with a road network where roads are given by polygonal areas.
- [Haunert and Sester, 2008]: compute centerlines of roads resp. extract the corresponding network graph using straight skeletons.



Roof construction

- We are given the footprint of a house as a polygon P and want to design a so-called hip roof for it.
  - All faces have the same slope.
  - There are no local minima within P, where rain accumulates.
  - [Aichholzer et al., 1995]: the terrain model gives us a solution.
- [Laycock and Day, 2003]: use heuristics to generate gable roofs, mansard roofs, gambrel roofs, and Dutch roofs.

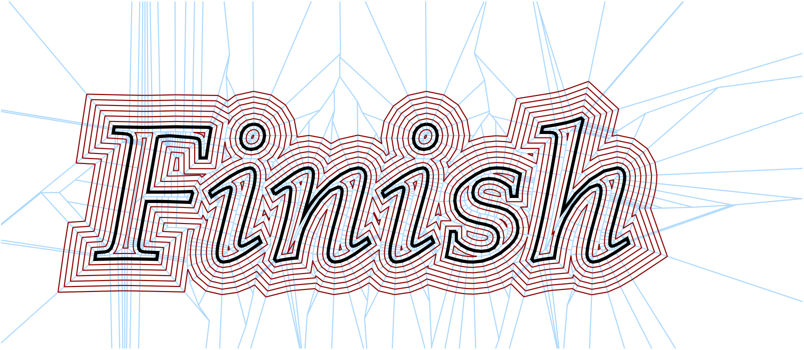


Terrain modeling

Similar problem to roof construction:

- We are given a river or a lake and want to model a mountain terrain in its neighborhood.
- [Aichholzer and Aurenhammer, 1998]: use the straight skeleton.

Figure : Left: terrain generated by BONE. Right: Actual photo from de.wikipedia.org, CC BY-SA 3.0 license, originator: Techcollector


Finish

Bibliography I

Aichholzer, O., Alberts, D., Aurenhammer, F., and Gärtner, B. (1995).
A novel type of skeleton for polygons.
J. Universal Comp. Sci., 1(12):752–761.

Aichholzer, O. and Aurenhammer, F. (1998).
Straight skeletons for general polygonal figures in the plane.
In Samoilenko, A., editor, *Voronoi's Impact on Modern Science, Book 2*, pages 7–21. Institute of Mathematics of the National Academy of Sciences of Ukraine, Kiev, Ukraine.

Haunert, J.-H. and Sester, M. (2008).
Area collapse and road centerlines based on straight skeletons.
GeoInformatica, 12:169–191.

Held, M. and Huber, S. (2009).
Topology-oriented incremental computation of Voronoi diagrams of circular arcs and straight-line segments.
Comp. Aided Design, 41(5):327–338.

Bibliography II

Laycock, R. G. and Day, A. M. (2003).

Automatically generating large urban environments based on the footprint data of buildings.

In *Proc. 8thACM Symp. on Solid Mod. & Appl. (SM '03)*, pages 346–351, Seattle, Washington, USA.