Skeleton Structures in Computational Geometry
An introduction with GIS in mind

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Motivation

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- We would like to have a geometric structure that
  - lets us easily identify “bottlenecks” of $P$ and
  - allows us to reuse known path-finding algorithms (on graphs).
- For short: we need information about the shape of $P$.
  - Skeleton structures can do that.
Generalized Voronoi diagrams

The Voronoi diagram of points in the Euclidean plane has been generalized in multiple ways:

- plane $\rightarrow$ higher dimensions
- points $\rightarrow$ straight-line segments, circular arcs, ...
- Euclidean $\rightarrow$ $L^k$-norms, convex distance functions, ...

In this talk: Voronoi diagram of points, straight-line segments and circular arcs in the Euclidean plane.
Generalized Voronoi diagrams

Given: set $S$ of **input sites**, i.e., points, straight-line segments, circular arcs, not intersecting in their relative interior.

- Plane tessellated into **cells around input sites**.
  - Points within the cell of site $i$ are **closer to** $i$ than to all other sites.
  - Bisectors are parabolic/elliptic arcs.
- $\mathcal{V}(S)$ is the Voronoi diagram of $S$. That is, $\mathcal{V}(S)$ consists of the boundaries of all cells.
Generalized Voronoi diagrams: details

What if sites touch?

Any point in the shaded area is equidistant to both segments.
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- Endpoints of segments and arcs are input sites, too.
- Voronoi cell of a site is restricted to “cone of influence”.
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Precise definition in [Held and Huber, 2009].
Voronoi diagram of a shape

We are given a simple polygon $P$.

- $P$ consists of vertices and edges $\rightarrow$ take them as the set $S$ of input sites.
- Short-hand notation: $\mathcal{V}(P)$ is the resulting Voronoi diagram, i.e., $\mathcal{V}(S)$.
  - $P$ is tessellated into Voronoi cells.
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  - Sometimes $\mathcal{V}(P)$ is considered to be restricted to $P$.
  - Sometimes, $P$ may have holes and its boundary may also comprise circular arcs.
Finding bottlenecks of shapes

- The **clearance disk** $C(p)$ is the largest disk within $P$ centered at the point $p$.
  - Its radius is the **clearance radius**.

![Diagram](image-url)
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- **Bottleneck**: a Voronoi node with locally minimal clearance radius.
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- Project $p$ resp. $q$ to points $p'$ resp. $q'$ on Voronoi edges.
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Can we move $V$ form $p$ to $q$ within $P$?

- Compute $\mathcal{V}(P)$. Consider the bottlenecks, whose clearance disk is smaller than $V$, removed.
- Project $p$ resp. $q$ to points $p'$ resp. $q'$ on Voronoi edges.
- Find a path from $p'$ to $q'$ on $\mathcal{V}(P)$ using ordinary graph algorithms.
Medial axis

We are given a shape $P$.

- The medial axis $\mathcal{M}(P)$ consists of those points $p$ within $P$ whose minimum distance to the boundary of $P$ is assumed at two or more boundary points.
- That is, the clearance disk at $p$ touches $P$ at two or more points.
- Hence, $\mathcal{M}(P) \subseteq \mathcal{V}(P)$.
  - $\mathcal{M}(P)$ is easily extracted from $\mathcal{V}(P)$. 

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Maximum inscribed circle

What is the largest disk we can place in a shape $P$?
- Kind of complementary problem to bottleneck detection.

Algorithm:
Maximum inscribed circle

What is the largest disk we can place in a shape $P$?

- Kind of complementary problem to bottleneck detection.

Algorithm:

- Take the Voronoi node with largest clearance radius.
$P$ equals the union of all clearance disks placed on $\mathcal{M}(P)$.

- $\mathcal{M}(P)$ and the clearance radius function $r(.)$ on $\mathcal{M}(P)$ can together reconstruct $P$.

- Besides "thickness", $\mathcal{M}(P)$ contains the essential information of the "shape" of $P$.
  - For instance, holes in $P$ correspond to cycles in $\mathcal{M}(P)$.
Reconstruction property, topological view

We are given networks of rivers by their polygonal shapes.

- How to find the center-lines of the rivers?
- How to find the main branches?
- How to prune small creeks?
- Which river is connected with which?
- How to remove details from a map, i.e., collapse small rivers or streets to a line?
Minkowski-sum and Minkowski-difference

The **Minkowski-sum** $A \oplus B$ of two sets $A$ and $B$ is the union of all $B$ moved by a vector $v \in A$. Note that $B \oplus A = A \oplus B$.

The **Minkowski-difference** $A \ominus B$ of two sets $A$ and $B$ is the largest set such that its Minkowski-sum with $B$ is contained in $A$. 
Offsetting

Let $D_r$ denote the disk with radius $r$ and the origin as center.

- How to compute all points inside/outside of $P$ that have a distance of exactly (or at most) $r$? That is, **how to compute** $P \oplus D_r$ resp. $P \ominus D_r$?
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**Collision-free paths:** Note that there is a collision-free path for $V$ from $p$ to $q$ if there is any path from $p$ to $q$ within $P \ominus V$. 
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Offsets have many **more applications:** computing tolerance zones, tool-paths in NC-machining, buffering in maps, thinning or extruding shapes, ...
The Voronoi diagram of a simple polygon $P$ can be characterized by means of interference patterns of offset segments $\rightarrow$ wavefront propagation.
Straight skeletons: Definition

Suppose that we do not like circular arcs in the offset curves.

- Replace circular arcs by straight-line caps.
- Each wavefront edge is parallel to an edge of $P$ and moves with equal speed.

The straight skeleton is defined by interference patterns of “mitered-offset curves”.

(a) Voronoi diagram

(b) straight skeleton
Straight skeletons: Definition

- Topological changes (events) during the wavefront propagation:
  - **Edge event**: a wavefront edge shrinks to zero length.
  - **Split event**: a reflex wavefront vertex splits another wavefront edge.
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- Straight skeleton $S(P)$: set of loci of all wavefront vertices.
  - Each **node** of $S(P)$ is the locus of an event.
  - Each **arc** is on the bisector of polygon edges.
  - Each **face** is swept out by the wavefront edge emanated by an input edge.
Straight skeleton of a PSLG

- PSLG: planar straight-line graph, i.e., a bunch of straight-line segments that do not intersect in their relative interior.
- [Aichholzer and Aurenhammer, 1998]: straight skeleton $S(G)$ of a PSLG $G$
  - Each input edge sends out two parallel wavefront copies.
  - Each terminal vertex sends out an additional wavefront edge.
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Consider the wavefront propagation in three-dimensional space-time, with the $z$-axis representing time.

An isoline of the resulting $T(G)$ corresponds to the wavefront at some point in time.

Projecting the valleys and ridges onto $\mathbb{R}^2$ gives us $S(G)$ again.

Knowing $T(G)$ is equivalent to knowing $S(G)$. 
Area collapsing

- Let us consider a map with a river given as a polygonal area.
- How to reduce the level of detail by collapsing the river’s area to a line?
- [Haunert and Sester, 2008]:
  - Compute $S(P)^1$, which tessellates $P$ into faces.
  - To each edge $e$ of $P$ belongs a face $f(e)$ and each edge $e$ also belongs to a neighboring polygon $Q$.
  - Add $f(e)$ to $Q$.

\[\text{\footnotesize\cite{Haunert2008}}\]

\[^1\text{Actually, weighted straight skeletons are employed.}\]
Centerlines of roads

Similar problem to area collapsing:

- Let us consider a map with a road network where roads are given by polygonal areas.
- [Haunert and Sester, 2008]: compute centerlines of roads resp. extract the corresponding network graph using straight skeletons.
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Roof construction

- We are given the footprint of a house as a polygon $P$ and want to design a so-called hip roof for it.
  - All faces have the same slope.
  - There are no local minima within $P$, where rain accumulates.
  - [Aichholzer et al., 1995]: the terrain model gives us a solution.
- [Laycock and Day, 2003]: use heuristics to generate gable roofs, mansard roofs, gambrel roofs, and Dutch roofs.
Terrain modeling

Similar problem to roof construction:

- We are given a river or a lake and want to model a mountain terrain in its neighborhood.
- [Aichholzer and Aurenhammer, 1998]: use the straight skeleton.

**Figure**: Left: terrain generated by Bone. Right: Actual photo from de.wikipedia.org, CC BY-SA 3.0 license, originator: Techcollector


Automatically generating large urban environments based on the footprint data of buildings.
In *Proc. 8th ACM Symp. on Solid Mod. & Appl. (SM '03)*, pages 346–351, Seattle, Washington, USA.