

Straight Skeletons By Means of Voronoi Diagrams

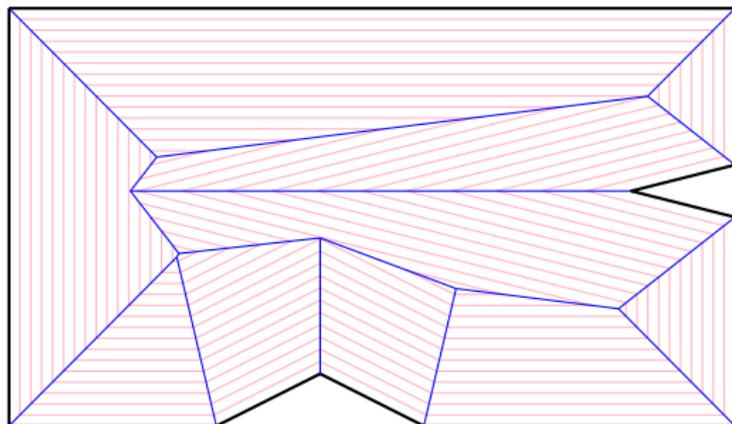
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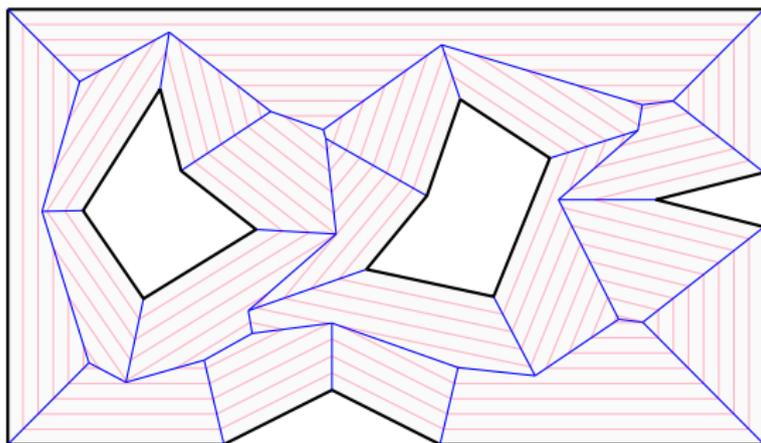
May 20, 2014

Straight skeletons



- ▶ Wavefront propagation:
 - ▶ At time t the wavefront $\mathcal{W}_S(t)$ forms a mitered offset.
 - ▶ Events: structural changes of the wavefront over time.
- ▶ $S(P)$ is the set of loci traced out by vertices of $\mathcal{W}_S(t)$.

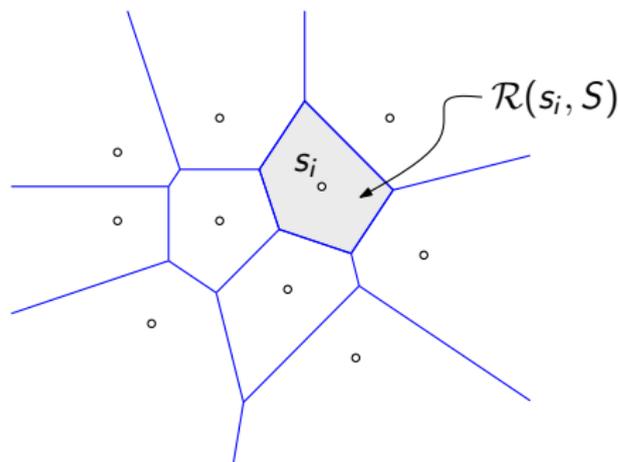
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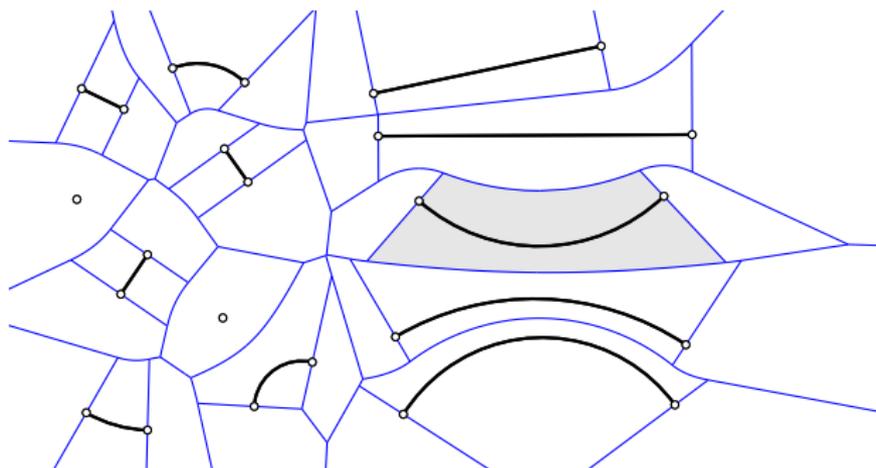
Voronoi diagrams

- ▶ Given:
 - ▶ A metric space $(\mathbb{R}^d, \|\cdot\|)$.
 - ▶ A finite set $S = \{s_1, \dots, s_n\}$ of *input sites*.
- ▶ Voronoi region $\mathcal{R}(s_i, S) = \{q \in \mathbb{R}^d : \|q - s_i\| \leq \|q - s_j\|, 1 \leq j \leq n\}$.
- ▶ Voronoi diagram $\mathcal{V}(S) = \bigcup_{i=1}^n \partial \mathcal{R}(s_i, S)$.



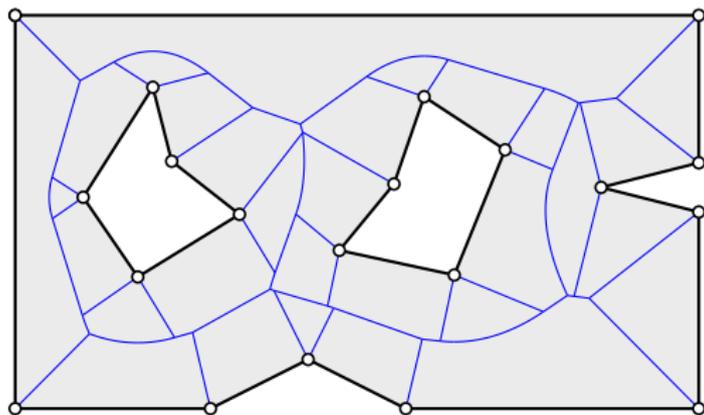
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Voronoi diagram of a polygon

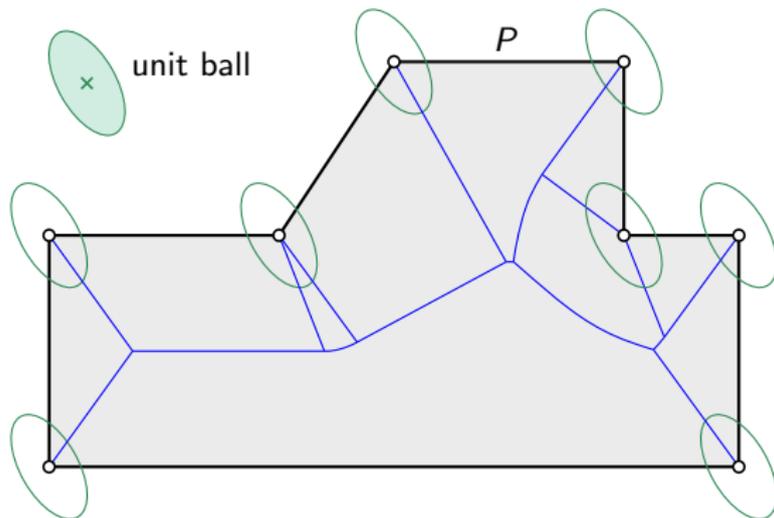
- ▶ Given: A polygon (with holes) P .
- ▶ Interpret the vertices and edges of P as input sites S .
- ▶ $\mathcal{V}(P) = \mathcal{V}(S) \cap P$.



- ▶ $\mathcal{V}(P)$ tessellates P into Voronoi regions.

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Straight skeleton versus Voronoi diagram

- ▶ The straight skeleton does not fit into the Abstract Voronoi Diagram framework of Klein.
- ▶ Computing $\mathcal{S}(P)$ is \mathcal{P} -complete.
- ▶ The straight skeleton is prone to non-local effects.
- ▶ $\mathcal{S}(P)$ changes discontinuously when moving vertices of P .

TL'DR: The straight skeleton is fundamentally different from the Voronoi diagram.

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On the other hand:

- ▶ P rectilinear, $(\mathbb{R}^2, \|\cdot\|_\infty)$: $\mathcal{V}(P) = \mathcal{S}(P)$.
- ▶ P 's reflex vertices "rounded", $(\mathbb{R}^2, \|\cdot\|_2)$: $\mathcal{V}(P) = \mathcal{S}(P)$.

Question

Under which circumstances is $\mathcal{V}(P) = \mathcal{S}(P)$?

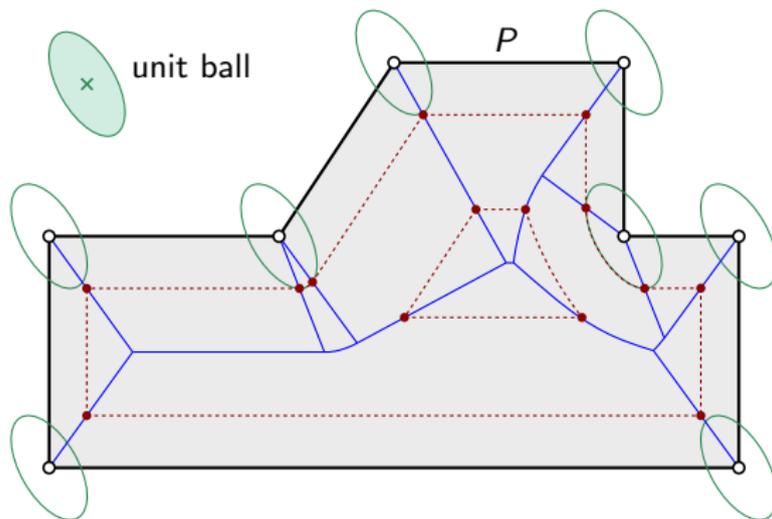
Why?

Best of both worlds:

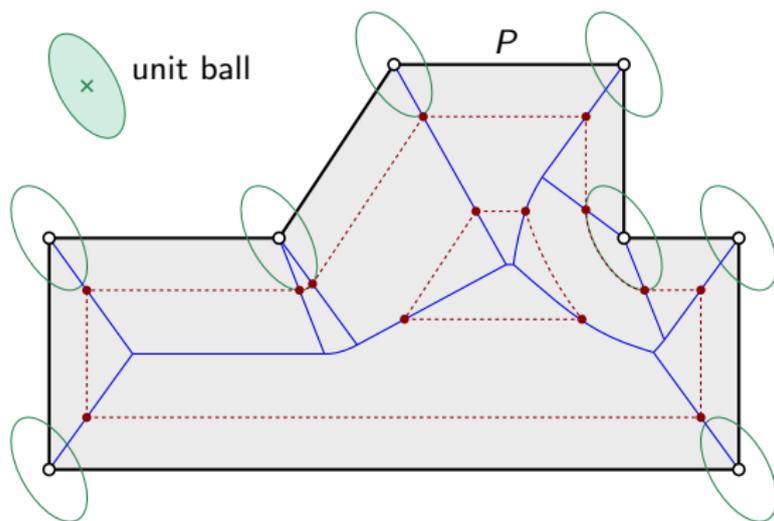
- ▶ Optimal algorithms for $\mathcal{V}(P)$ in \mathbb{R}^2 known, but not for $\mathcal{S}(P)$.
- ▶ Definition for $\mathcal{S}(P)$ in \mathbb{R}^3 is a pain, but not for $\mathcal{V}(P)$.
- ▶ $\mathcal{S}(P)$ comprises piecewise-linear features only, but $\mathcal{V}(P)$ does not.
- ▶ $\mathcal{V}(P)$ changes continuously, $\mathcal{S}(P)$ does not, et cetera.

Voronoi diagrams by means of wavefronts

- ▶ $X, Y \subseteq \mathbb{R}^d$:
 - ▶ $X \oplus Y = \{x + y : x \in X, y \in Y\}$.
 - ▶ $X \ominus Y = \{z \in \mathbb{R}^d : \{z\} \oplus Y \subseteq X\}$.
- ▶ Unit ball $B = \{x \in \mathbb{R}^d : \|x\| \leq 1\}$.
- ▶ Minkowski offset $\mathcal{W}_Y(t) = \partial(P \ominus tB)$.



Voronoi diagrams by means of wavefronts



$$\begin{aligned}\mathcal{W}_V(t) &= \partial(P \ominus tB) \\ &= P \cap \partial(\partial P \oplus tB) \\ &= P \cap \bigcup_{\text{face } s \text{ of } \partial P} \mathcal{R}(s, P) \cap \partial(s \oplus t \cdot B)\end{aligned}$$

Voronoi diagrams by means of wavefronts

- ▶ $\mathcal{V}(P)$ is the interference pattern of the wavefront \mathcal{W}_V .
- ▶ The norm $\|\cdot\|$ can be specified by a unit ball B :
 - ▶ $\|x\|_B = \inf\{t \geq 0: x \in tB\}$ for any $x \in \mathbb{R}^d$.

Question

For which unit balls B and for which input shapes P is $\mathcal{W}_S(t) = \mathcal{W}_V(t)$ for all $t \geq 0$?

Proper unit balls

$\|\cdot\|_B$ shall be a norm:

- ▶ B needs to be convex and o -symmetric.

$\mathcal{W}_S(t)$ has a piecewise-linear geometry.

- ▶ $\partial(P \ominus tB)$ comprises features of P and B .
- ▶ For $\mathcal{W}_S(t) = \mathcal{W}_V(t)$, B needs to be polyhedral.

At least for $P = B$ we would like that $\mathcal{W}_S(t) = \mathcal{W}_V(t)$.

- ▶ $\mathcal{W}_V(t) = (1 - t)B$.
- ▶ All facets of \mathcal{W}_V reach o at time 1.
- ▶ All facets of \mathcal{W}_S need to reach o at time 1.
- ▶ All facets of B have distance 1 to o .
- ▶ We call such a B *isotropic*.

Proper unit balls

Definition

A proper unit ball is a convex, o -symmetric, isotropic polyhedron.

Lemma

For a proper unit ball B and any $v \in \mathbb{R}^d$ it holds that $\|v\|_2 \geq \|v\|_B$, and equality holds exactly when v is a normal vector of a facet of B .

Proper input shapes

Definition

A (d -dimensional) *input shape* P is a connected, compact set in \mathbb{R}^d whose boundary forms a polyhedral surface that constitutes an orientable $(d - 1)$ -manifold.

Definition

A face f of P of dimension at most $d - 2$ is called *reflex* if for any point p in the relative interior of f and for any Euclidean ball O , which is centered at p and has sufficiently small but positive radius, $O \setminus P$ is contained in a half-space whose boundary supports p .

Corresponding facets

For a facet f of P let $n(f)$ be the normal vector of f pointing to the interior.

Lemma

Every facet f of P has a corresponding facet f^B of B that has $n(f)$ as the outer normal vector, unless $\mathcal{W}_V(\varepsilon) \neq \mathcal{W}_S(\varepsilon)$ for some $\varepsilon > 0$.

Two-dimensional input shapes

The last lemma says:

- ▶ For every edge e of P there is a corresponding edge e^B of B .

Lemma

Let v be a reflex vertex of P with incident edges e_1 and e_2 . Then there is a corresponding vertex v^B of B that is incident to e_1^B and e_2^B , unless $\mathcal{W}_V(\varepsilon) \neq \mathcal{W}_S(\varepsilon)$ for some $\varepsilon > 0$.

The existence of corresponding edges and reflex vertices is necessary for $\mathcal{W}_V(t) = \mathcal{W}_S(t)$.

Two-dimensional input shapes

Definition

A *proper* input shape P w.r.t. a proper unit ball B in \mathbb{R}^2 is a polygon with holes such that

- (I1) for each edge e of P there is a corresponding edge e^B of B whose outer normal vector is $n(e)$ and
- (I2) for each reflex vertex v of P , incident to edges e_1 and e_2 , there is a corresponding vertex v^B of B that is incident to e_1^B and e_2^B .

Theorem

For a proper input shape P w.r.t. a proper unit ball B in \mathbb{R}^2 it holds that $\mathcal{W}_S(t) = \mathcal{W}_V(t)$ for all $t \geq 0$.

Higher-dimensional input shapes

We know: each facet f of P has a corresponding facet f^B in B .

For $d = 2$: a proper input shape looks locally the same as a unit ball at non-convex features.

- ▶ For $d > 2$ we have a larger “diversity” of non-convexity.
- ▶ For $(d - 2)$ -dimensional faces the situation is still simpler.

Lemma

Let P be an input shape in \mathbb{R}^d , where $d \geq 2$. For each reflex $(d - 2)$ -dimensional face e of P , which is incident to facets f_1 and f_2 , it holds that $f_1^B \cap f_2^B \neq \emptyset$, unless $\mathcal{W}_V(\varepsilon) \neq \mathcal{W}_S(\varepsilon)$ for some $\varepsilon > 0$.

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From $f_1^B \cap f_2^B \neq \emptyset$ it does not follow that $f_1^B \cap f_2^B$ forms a $(d - 2)$ -dimensional face of B !

Proper input shapes

Definition

An input shape P in \mathbb{R}^d is called *proper* w.r.t. a proper unit ball B if

- (I1) for each facet f of P there is a corresponding facet f^B of B whose outer normal vector is $n(f)$ and
- (I2) for all points p on all facets f of P , there is a point p' such that $\inf_{q \in P} \|p' - q\|_B = \|p' - p\|_B > 0$ and $p \in \text{relint}_f(f \cap (p' + \|p' - p\|_B \partial B))$.

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Lemma

For any proper input shape P w.r.t. B there is a finite point set S , with $P \cap S = \emptyset$, and some $\varepsilon > 0$ such that $\partial P \subseteq \partial(S \oplus \varepsilon B)$.

Corresponding reflex faces

Lemma

Let e be a reflex face of dimension k of a proper input shape P in \mathbb{R}^d , where $0 \leq k \leq d - 2$. Then for any point $p \in \text{relint } e$ there is a point $p^B \in \partial B$ such that for some $\varepsilon, \varepsilon' > 0$ the sets $\partial P \cap (p + \varepsilon O)$ and $\partial B \cap (p^B + \varepsilon' O)$ are homothetic, where O denotes the Euclidean unit ball. In particular, to e corresponds a k -dimensional face e^B of B with $p^B \in \text{relint } e^B$.

Theorem

For a proper input shape P w.r.t. a proper unit ball B in \mathbb{R}^d it holds that $\mathcal{W}_S(t) = \mathcal{W}_V(t)$ for all $t \geq 0$.

Approximation by proper input shapes

Lemma

For any input shape P and any $\varepsilon > 0$ there is proper input shape P' with $P \subseteq P' \subseteq P \oplus \varepsilon O$.