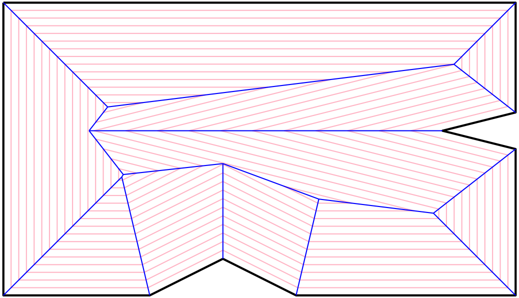
Straight Skeletons By Means of Voronoi Diagrams

Stefan Huber[1] Oswin Aichholzer[2] Thomas Hackl[2] Birgit Vogtenhuber[2]

[1]Institute of Science and Technology Austria
[2]Institute for Software Technology, Graz University of Technology, Austria

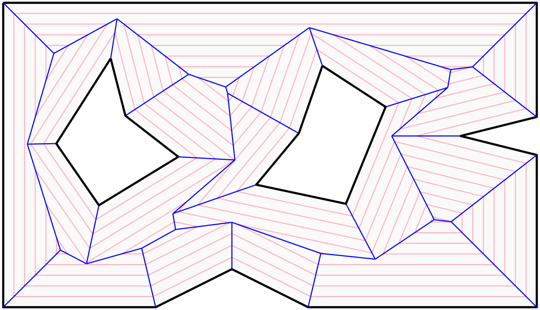
May 20, 2014

Straight skeletons



- Wavefront propagation:
  - At time t the wavefront $\mathcal{W}_{\mathcal{S}}(t)$ forms a mitered offset.
  - Events: structural changes of the wavefront over time.
- $\mathcal{S}(P)$ is the set of loci traced out by vertices of $\mathcal{W}_{\mathcal{S}}(t)$.

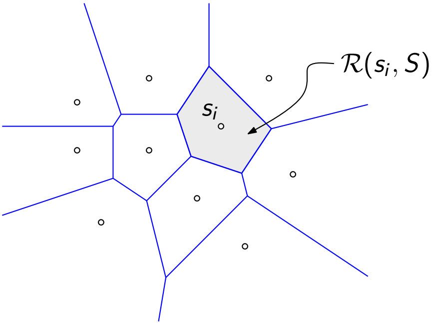
Straight skeletons



- Wavefront propagation:
  - At time t the wavefront $\mathcal{W}_{\mathcal{S}}(t)$ forms a mitered offset.
  - Events: structural changes of the wavefront over time.
- $\mathcal{S}(P)$ is the set of loci traced out by vertices of $\mathcal{W}_{\mathcal{S}}(t)$.

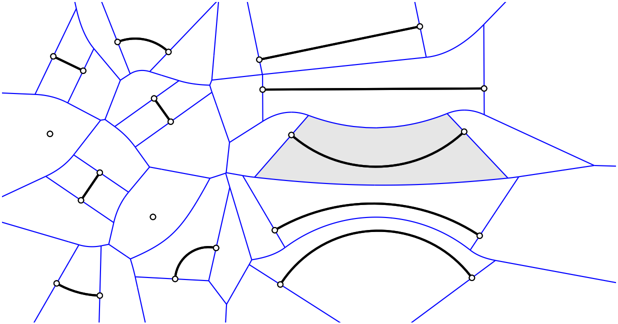
Voronoi diagrams

- Given:
  - A metric space $(\mathbb{R}^d, \|.\|)$.
  - A finite set $S = \{s_1, \ldots, s_n\}$ of *input sites*.
- Voronoi region $\mathcal{R}(s_i, S) = \{q \in \mathbb{R}^d \colon \|q - s_i\| \leq \|q - s_j\|, 1 \leq j \leq n\}$.
- Voronoi diagram $\mathcal{V}(S) = \bigcup_{i=1}^{n} \partial\,\mathcal{R}(s_i, S)$.



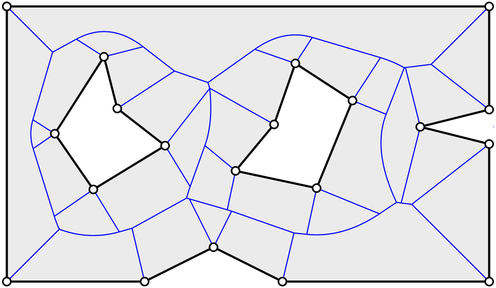
Voronoi diagrams

- Given:
  - A metric space $(\mathbb{R}^d, \|.\|)$.
  - A finite set $S = \{s_1, \ldots, s_n\}$ of *input sites*.
- Voronoi region $\mathcal{R}(s_i, S) = \{q \in \mathbb{R}^d \colon \|q - s_i\| \leq \|q - s_j\|, 1 \leq j \leq n\}$.
- Voronoi diagram $\mathcal{V}(S) = \bigcup_{i=1}^{n} \partial\,\mathcal{R}(s_i, S)$.



Voronoi diagram of a polygon

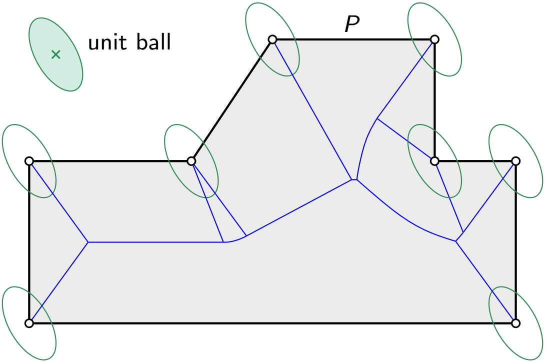
- Given: A polygon (with holes) P.
- Interpret the vertices and edges of P as input sites S.
- $\mathcal{V}(P) = \mathcal{V}(S) \cap P$.



- $\mathcal{V}(P)$ tessellates P into Voronoi regions.

Voronoi diagram of a polygon

- Given: A polygon (with holes) P.
- Interpret the vertices and edges of P as input sites S.
- $\mathcal{V}(P) = \mathcal{V}(S) \cap P$.



- $\mathcal{V}(P)$ tessellates P into Voronoi regions.

Straight skeleton versus Voronoi diagram

- The straight skeleton does not fit into the Abstract Voronoi Diagram framework of Klein.
- Computing $\mathcal{S}(P)$ is $\mathcal{P}$-complete.
- The straight skeleton is prone to non-local effects.
- $\mathcal{S}(P)$ changes discontinuously when moving vertices of P.

TL'DR: The straight skeleton is fundamentally different from the Voronoi diagram.

Straight skeleton versus Voronoi diagram

- The straight skeleton does not fit into the Abstract Voronoi Diagram framework of Klein.
- Computing $\mathcal{S}(P)$ is $\mathcal{P}$-complete.
- The straight skeleton is prone to non-local effects.
- $\mathcal{S}(P)$ changes discontinuously when moving vertices of P.

TL'DR: The straight skeleton is fundamentally different from the Voronoi diagram.

On the other hand:

- P rectilinear, $(\mathbb{R}^2, \|.\|_\infty)$: $\mathcal{V}(P) = \mathcal{S}(P)$.
- P's reflex vertices "rounded", $(\mathbb{R}^2, \|.\|_2)$: $\mathcal{V}(P) = \mathcal{S}(P)$.

Question

Under which circumstances is $\mathcal{V}(P) = \mathcal{S}(P)$?

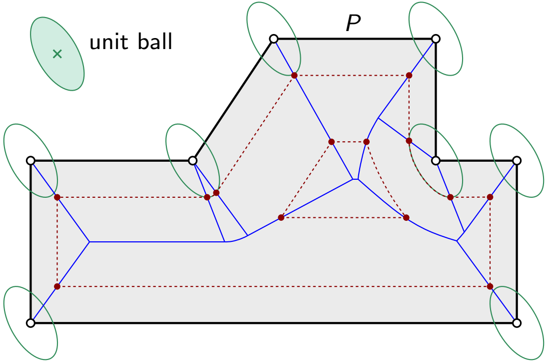
Why?

Best of both worlds:

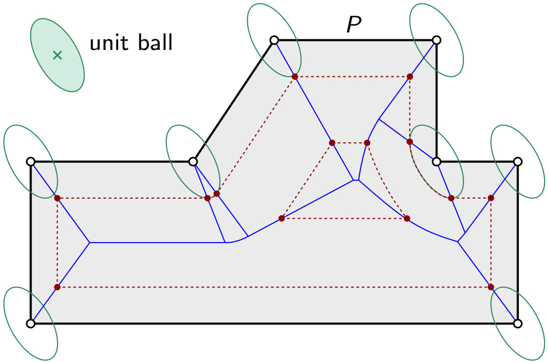
- Optimal algorithms for $\mathcal{V}(P)$ in $\mathbb{R}^2$ known, but not for $\mathcal{S}(P)$.
- Definition for $\mathcal{S}(P)$ in $\mathbb{R}^3$ is a pain, but not for $\mathcal{V}(P)$.
- $\mathcal{S}(P)$ comprises piecewise-linear features only, but $\mathcal{V}(P)$ does not.
- $\mathcal{V}(P)$ changes continuously, $\mathcal{S}(P)$ does not, et cetera.

Voronoi diagrams by means of wavefronts

- $X, Y \subseteq \mathbb{R}^d$:
  - $X \oplus Y = \{x + y \colon x \in X, y \in Y\}$.
  - $X \ominus Y = \{z \in \mathbb{R}^d \colon \{z\} \oplus Y \subseteq X\}$.
- Unit ball $B = \{x \in \mathbb{R}^d \colon \|x\| \leq 1\}$.
- Minkowski offset $\mathcal{W}_{\mathcal{V}}(t) = \partial(P \ominus tB)$.



Voronoi diagrams by means of wavefronts



$$\begin{aligned} \mathcal{W}_{\mathcal{V}}(t) &= \partial(P \ominus tB) \\ &= P \cap \partial(\partial P \oplus tB) \\ &= P \cap \bigcup_{\text{face } s \text{ of } \partial P} \mathcal{R}(s, P) \cap \partial(s \oplus t \cdot B) \end{aligned}$$

Voronoi diagrams by means of wavefronts

- $\mathcal{V}(P)$ is the interference pattern of the wavefront $\mathcal{W}_{\mathcal{V}}$.
- The norm $\|.\|$ can be specified by a unit ball B:
  - $\|x\|_B = \inf\{t \geq 0 \colon x \in tB\}$ for any $x \in \mathbb{R}^d$.

Question

For which unit balls B and for which input shapes P is $\mathcal{W}_{\mathcal{S}}(t) = \mathcal{W}_{\mathcal{V}}(t)$ for all $t \geq 0$?

Proper unit balls

$\|.\|_B$ shall be a norm:

- B needs to be convex and o-symmetric.

$\mathcal{W}_\mathcal{S}(t)$ has a piecewise-linear geometry.

- $\partial(P \ominus tB)$ comprises features of P and B.
- For $\mathcal{W}_\mathcal{S}(t) = \mathcal{W}_\mathcal{V}(t)$, B needs to be polyhedral.

At least for $P = B$ we would like that $\mathcal{W}_\mathcal{S}(t) = \mathcal{W}_\mathcal{V}(t)$.

- $\mathcal{W}_\mathcal{V}(t) = (1 - t)B$.
- All facets of $\mathcal{W}_\mathcal{V}$ reach o at time 1.
- All facets of $\mathcal{W}_\mathcal{S}$ need to reach o at time 1.
- All facets of B have distance 1 to o.
- We call such a B *isotropic*.

Proper unit balls

Definition

A proper unit ball is a convex, *o*-symmetric, isotropic polyhedron.

Lemma

For a proper unit ball B and any $v \in \mathbb{R}^d$ it holds that $\|v\|_2 \geq \|v\|_B$, and equality holds exactly when v is a normal vector of a facet of B.

Proper input shapes

Definition

A (d-dimensional) *input shape* P is a connected, compact set in $\mathbb{R}^d$ whose boundary forms a polyhedral surface that constitutes an orientable $(d-1)$-manifold.

Definition

A face f of P of dimension at most $d-2$ is called *reflex* if for any point p in the relative interior of f and for any Euclidean ball O, which is centered at p and has sufficiently small but positive radius, $O \setminus P$ is contained in a half-space whose boundary supports p.

Corresponding facets

For a facet f of P let $n(f)$ be the normal vector of f pointing to the interior.

Lemma

Every facet f of P has a corresponding facet f^B of B that has $n(f)$ as the outer normal vector, unless $\mathcal{W}_{\mathcal{V}}(\varepsilon) \neq \mathcal{W}_{\mathcal{S}}(\varepsilon)$ for some $\varepsilon > 0$.

Two-dimensional input shapes

The last lemma says:

- For every edge e of P there is a corresponding edge e^B of B.

Lemma

Let v be a reflex vertex of P with incident edges e_1 and e_2. Then there is a corresponding vertex v^B of B that is incident to e_1^B and e_2^B, unless $\mathcal{W}_{\mathcal{V}}(\varepsilon) \neq \mathcal{W}_{\mathcal{S}}(\varepsilon)$ for some $\varepsilon > 0$.

The existence of corresponding edges and reflex vertices is necessary for $\mathcal{W}_{\mathcal{V}}(t) = \mathcal{W}_{\mathcal{S}}(t)$.

Two-dimensional input shapes

Definition

A *proper* input shape P w.r.t. a proper unit ball B in $\mathbb{R}^2$ is a polygon with holes such that

(I1) for each edge e of P there is a corresponding edge e^B of B whose outer normal vector is $n(e)$ and

(I2) for each reflex vertex v of P, incident to edges e_1 and e_2, there is a corresponding vertex v^B of B that is incident to e_1^B and e_2^B.

Theorem

For a proper input shape P w.r.t. a proper unit ball B in $\mathbb{R}^2$ it holds that $\mathcal{W}_\mathcal{S}(t) = \mathcal{W}_\mathcal{V}(t)$ for all $t \geq 0$.

Higher-dimensional input shapes

We know: each facet f of P has a corresponding facet f^B in B.

For $d = 2$: a proper input shape looks locally the same as a unit ball at non-convex features.

- For $d > 2$ we have a larger "diversity" of non-convexity.
- For $(d-2)$-dimensional faces the situation is still simpler.

Lemma

Let P be an input shape in $\mathbb{R}^d$, where $d \geq 2$. For each reflex $(d-2)$-dimensional face e of P, which is incident to facets f_1 and f_2, it holds that $f_1^B \cap f_2^B \neq \emptyset$, unless $\mathcal{W}_{\mathcal{V}}(\varepsilon) \neq \mathcal{W}_{\mathcal{S}}(\varepsilon)$ for some $\varepsilon > 0$.

Higher-dimensional input shapes

We know: each facet f of P has a corresponding facet f^B in B.

For $d = 2$: a proper input shape looks locally the same as a unit ball at non-convex features.

- For $d > 2$ we have a larger "diversity" of non-convexity.
- For $(d-2)$-dimensional faces the situation is still simpler.

Lemma

Let P be an input shape in $\mathbb{R}^d$, where $d \geq 2$. For each reflex $(d-2)$-dimensional face e of P, which is incident to facets f_1 and f_2, it holds that $f_1^B \cap f_2^B \neq \emptyset$, unless $\mathcal{W}_\mathcal{V}(\varepsilon) \neq \mathcal{W}_\mathcal{S}(\varepsilon)$ for some $\varepsilon > 0$.

From $f_1^B \cap f_2^B \neq \emptyset$ it does not follow that $f_1^B \cap f_2^B$ forms a $(d-2)$-dimensional face of B!

Proper input shapes

Definition

An input shape P in $\mathbb{R}^d$ is called *proper* w.r.t. a proper unit ball B if

(I1) for each facet f of P there is a corresponding facet f^B of B whose outer normal vector is $n(f)$ and

(I2) for all points p on all facets f of P, there is a point p' such that $\inf_{q \in P} \|p' - q\|_B = \|p' - p\|_B > 0$ and $p \in \mathrm{relint}_f\,(f \cap (p' + \|p' - p\|_B\, \partial B))$.

Proper input shapes

Definition

An input shape P in $\mathbb{R}^d$ is called *proper* w.r.t. a proper unit ball B if

(I1) for each facet f of P there is a corresponding facet f^B of B whose outer normal vector is $n(f)$ and

(I2) for all points p on all facets f of P, there is a point p' such that $\inf_{q \in P} \|p' - q\|_B = \|p' - p\|_B > 0$ and $p \in \mathrm{relint}_f\,(f \cap (p' + \|p' - p\|_B\, \partial B))$.

Lemma

For any proper input shape P w.r.t. B there is a finite point set S, with $P \cap S = \emptyset$, and some $\varepsilon > 0$ such that $\partial P \subseteq \partial(S \oplus \varepsilon B)$.

Corresponding reflex faces

Lemma

Let e be a reflex face of dimension k of a proper input shape P in $\mathbb{R}^d$, where $0 \leq k \leq d-2$. Then for any point $p \in \operatorname{relint} e$ there is a point $p^B \in \partial B$ such that for some $\varepsilon, \varepsilon' > 0$ the sets $\partial P \cap (p + \varepsilon O)$ and $\partial B \cap (p^B + \varepsilon' O)$ are homothetic, where O denotes the Euclidean unit ball. In particular, to e corresponds a k-dimensional face e^B of B with $p^B \in \operatorname{relint} e^B$.

Theorem

For a proper input shape P w.r.t. a proper unit ball B in $\mathbb{R}^d$ it holds that $\mathcal{W}_{\mathcal{S}}(t) = \mathcal{W}_{\mathcal{V}}(t)$ for all $t \geq 0$.

Approximation by proper input shapes

Lemma

For any input shape P and any $\varepsilon > 0$ there is proper input shape P' with $P \subseteq P' \subseteq P \oplus \varepsilon O$.