

# Planar Matchings for Weighted Straight Skeletons

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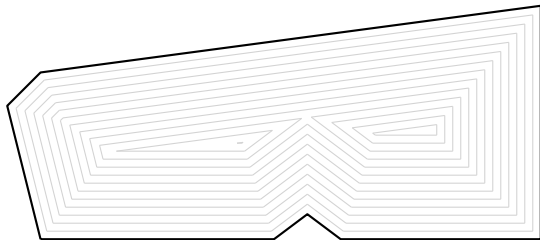
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Universität Salzburg, Austria

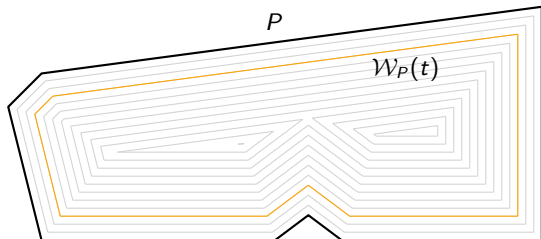
Geometry & Topology Seminar (IST Austria)  
November 11, 2014

# Straight skeletons: A brief introduction



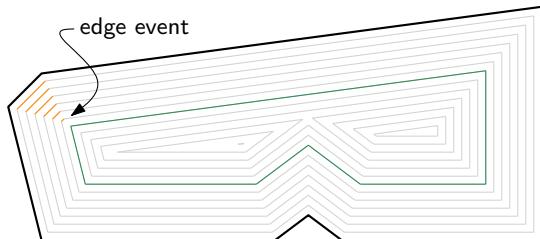
- ▶ Introduced by [Aichholzer et al., 1995].

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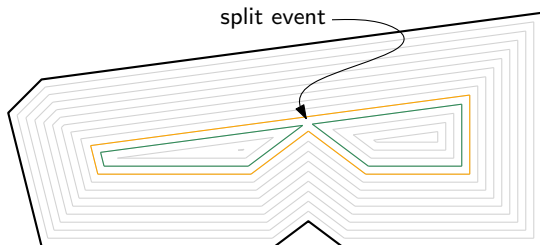
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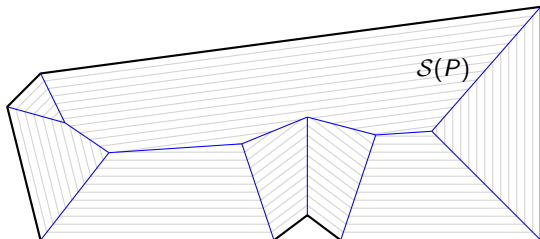
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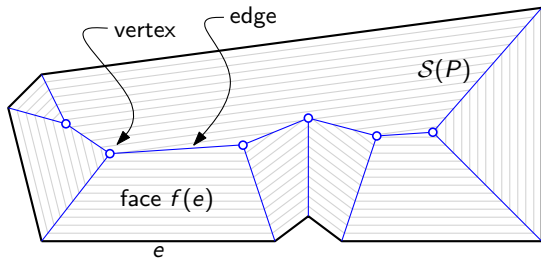
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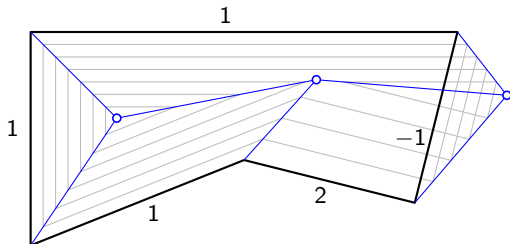
# Straight skeletons: A brief introduction



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# Weighted straight skeletons

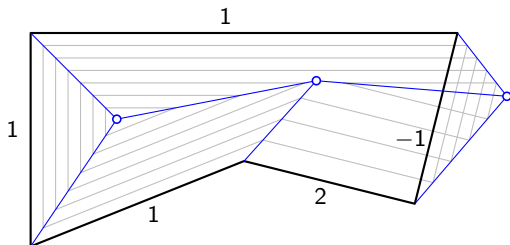
- ▶ First briefly studied in [Eppstein and Erickson, 1999].
- ▶ To every edge  $e$  of  $P$  a weight  $\sigma(e) \neq 0$  is assigned, its speed.





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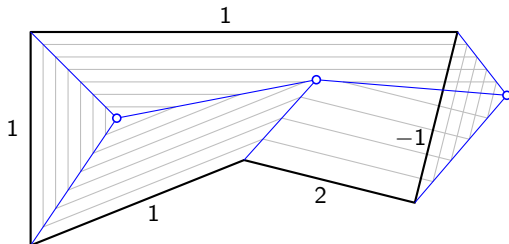


Weighted straight skeletons are “quite established”:

- ▶ Algorithms were published.
- ▶ Implementations are available.
- ▶ Used in theory & applications.

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**Still no rigorous definition is known so far!**

# Prior work

It was silently assumed that weights would not change much.

- ▶ Recently we showed that this is not quite true.

Property	Simple polygon			Polygon with holes		
	$\sigma \equiv 1$	$\sigma$ pos.	$\sigma$ arb.	$\sigma \equiv 1$	$\sigma$ pos.	$\sigma$ arb.
$S(P)$ is connected	✓	✓	✓	✓	✓	×
$S(P)$ has no crossing	✓	✓	×	✓	✓	×
$f(e)$ is monotone w.r.t. $e$	✓	×	×	✓	×	×
bd $f(e)$ is a simple polygon	✓	✓	×	✓	×	×
$\mathcal{T}(P)$ is $z$ -monotone <sup>1</sup>	✓	✓	×	✓	✓	×
$S(P)$ has $n(S(P)) - 1 + h$ edges	✓	✓	×	✓	✓	×
$S(P)$ is a tree	✓	✓	×			

Table: [Biedl et al., 2013, Biedl et al., 2015]

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<sup>1</sup> $\mathcal{T}(P) := \bigcup_{t>0} \mathcal{W}_P(t) \times \{t\}$ .

# Ambiguity of edge events

Ambiguity for parallel edges of different weights become adjacent.

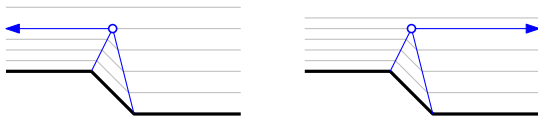


Figure: Resolution methods proposed in [Biedl et al., 2013, Biedl et al., 2015].

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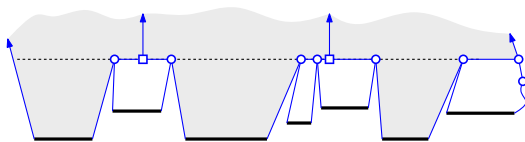
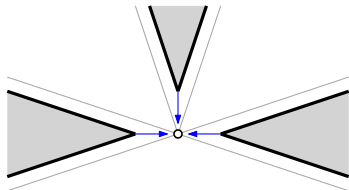


Figure: Resolution methods proposed in [Biedl et al., 2013, Biedl et al., 2015].

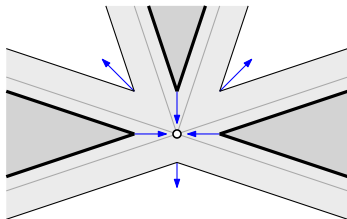
## Split events: pairing edges

The *standard scheme* works for unweighted straight skeletons.



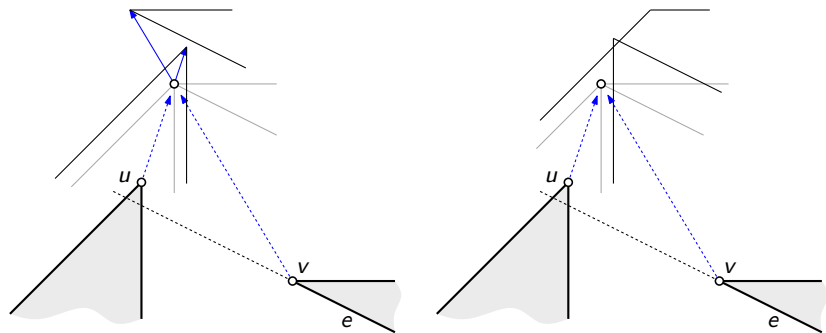
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Fundamental principle: Between events, the wavefront is a planar collection of wavefront polygons.

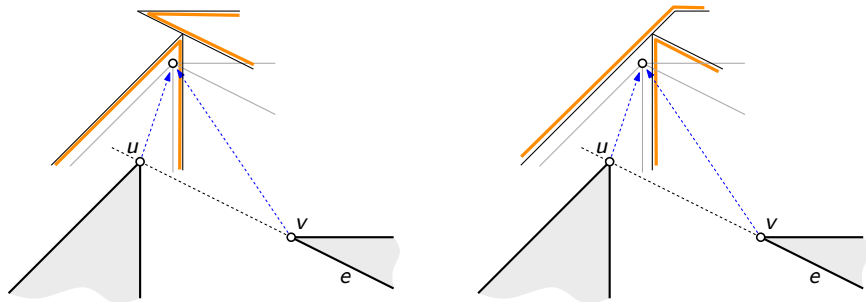
# Insufficiency of standard pairing technique



- ▶ The standard pairing technique can cause intersections.



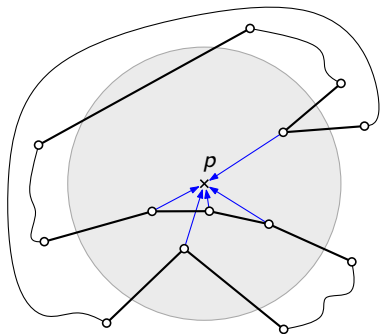
# Insufficiency of standard pairing technique



- ▶ The standard pairing technique can cause intersections.
- ▶ No pairing may exist that gives a planar wavefront.

# Split events with many vertices

How to handle this?



# Weak planarity

- ▶ Let  $\Phi$  denote the set of all straight-line embeddings  $\varphi: V \rightarrow \mathbb{R}^2$  of a planar graph  $G = (V, E)$ .
- ▶ The pair  $(\Phi, d(\cdot, \cdot)_\infty)$  constitutes a metric space, where  $d(\cdot, \cdot)_\infty: \Phi \times \Phi \rightarrow \mathbb{R}$  is defined by  $d(\varphi, \varphi')_\infty = \max_{v \in V} \|\varphi(v) - \varphi'(v)\|$ .
- ▶ Note that the set of planar straight-line embeddings is an open set.

## Definition

The set of *weakly-planar* embeddings of  $G$  is the topological closure of the set of planar embeddings of  $G$ .

## Corollary

*For every weakly-planar embedding  $\varphi$  and for every  $\varepsilon > 0$  there is a planar  $\varepsilon$ -perturbation  $\varphi'$  of  $\varphi$ , that is,  $d(\varphi, \varphi')_\infty < \varepsilon$ .*

# New fundamental principle

At all times, the wavefront is a weakly-planar collection of polygons.

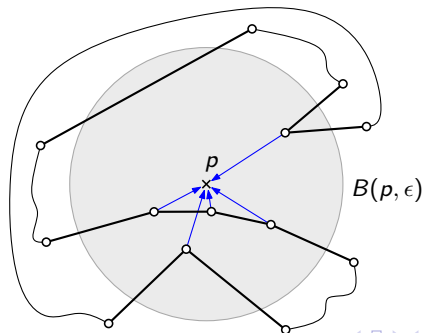
# Definition: Event

## Definition

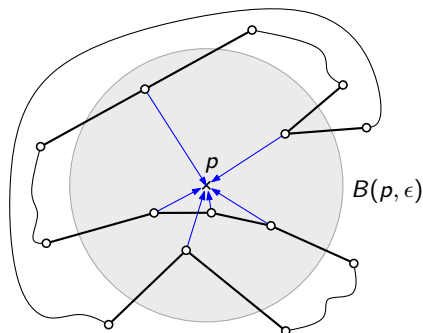
At location  $p$  and time  $t$  an *event* happens if at least two vertices meet at time  $t$  at  $p$  or if  $\exists \varepsilon_0 > 0 \forall 0 < \varepsilon < \varepsilon_0 \exists \delta > 0$  such that

- (i)  $\mathcal{W}(t') \cap B(p, \varepsilon)$  is non-empty and weakly-planar for  $t' \in [t - \delta, t]$  and
- (ii)  $\mathcal{W}(t') \cap B(p, \varepsilon)$  is non-empty and not weakly-planar for  $t' \in (t, t + \delta]$ .

We call the edges that meet  $p$  at time  $t$  the edges that are *involved* in the event.



## Pairing edges



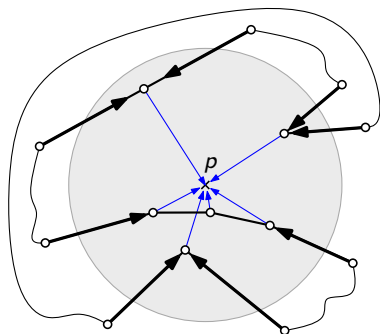
First:

- ▶ Remove collapsed edges.
- ▶ Split edges where both endpoints are outside  $B(p, \epsilon)$ .

Task: Find a pairing of remaining edges to **restore weak planarity**.

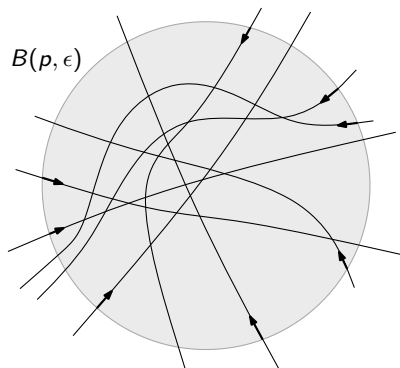
- ▶ Is this always possible? Uniquely?

# Directed pseudo-line arrangements



- ▶ We have  $k$  involved chains.
  - ▶ Hence,  $2k$  (non-collapsed) edges.
  - ▶ Assign direction to each edge.
- ▶ Consider directed supporting lines of edges, **after the event**.
  - ▶ Perturb a little to obtain general position.
  - ▶  $\rightarrow$  pseudo-line arrangement  $\mathcal{L}$  of directed pseudo-lines.
  - ▶ Obtain a “planar matching” of  $\mathcal{L}$  and revert perturbation.

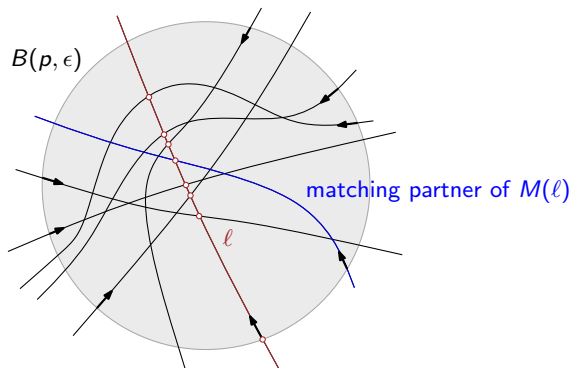
# Planar matchings



- ▶ Given: even number of directed pseudo lines.
- ▶ Every pair intersects, in a single unique point, within  $B(p, \epsilon)$ .

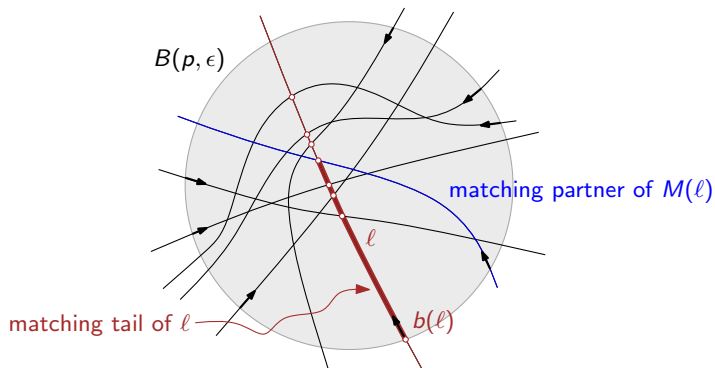


# Planar matchings



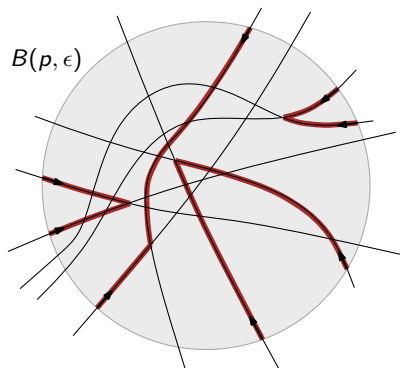
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# Planar matchings

## Theorem

*Every directed pseudo-line arrangement has a planar matching.*

# Stable roommates

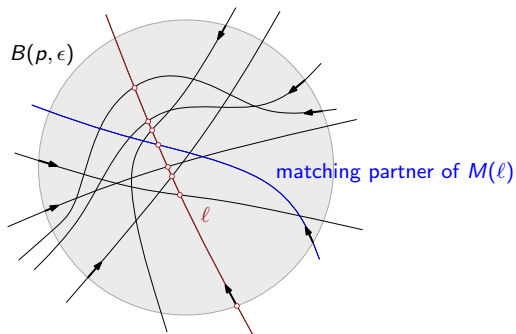
- ▶ Given: elements  $\mathcal{A} = \{a_1, \dots, a_N\}$ , with  $N$  even.
- ▶ Each element  $a_i$  has a preference list (**ranking**) of the other elements:
  - ▶ Complete: Every element of  $\mathcal{A}$  is contained in each ranking, and  $a_i$  prefers  $a_j$  least.
  - ▶ Strict: The ranking is a strict order.
- ▶ Consider a matching  $M$  on  $\mathcal{A}$ .
  - ▶ Blocking pair  $\{a_i, a_j\}$ : they prefer each other over their matching partners.
  - ▶  $M$  is **stable** if there is no blocking pair.
- ▶ Stable roommates problem: Find a stable matching.
  - ▶ Not every instance of the stable roommates problem has a solution!

# Stable roommates and planar matchings

- ▶ We are given a directed pseudo-line arrangement  $\mathcal{L}$ .
  - ▶ Intersection order gives the ranking.

## Lemma

$\mathcal{L}$  has a planar matching if and only if there is a stable matching.



# Stable partitions

- ▶ A **partition** is a permutation  $\pi$  of  $\ell_1, \dots, \ell_N$ .
  - ▶ Decompose  $\pi$  into cycles.
- ▶ A cycle of size  $\geq 3$  is a **semi-party** if each  $\ell$  in it prefers  $\pi(\ell)$  over  $\pi^{-1}(\ell)$ .
  - ▶ In a **semi-party partition** all cycles of size  $\geq 3$  are semi-parties.
- ▶ **Party-blocking pair**  $\{\ell_i, \ell_j\}$ : they prefer each other over  $\pi^{-1}(\ell_i)$  and  $\pi^{-1}(\ell_j)$ .
  - ▶ A **stable partition** is a semi-party partition with no party-blocking pair.

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## Theorem ([Tan and Hsueh, 1995])

1. *There is a stable partition, and it can be found in polynomial time.*
2. *There is a stable matching if and only if there is a stable partition with no cycles of odd size.*

## Theorem

*There are no parties of odd size for directed pseudo-line arrangements.*



# Odd parties do not exist

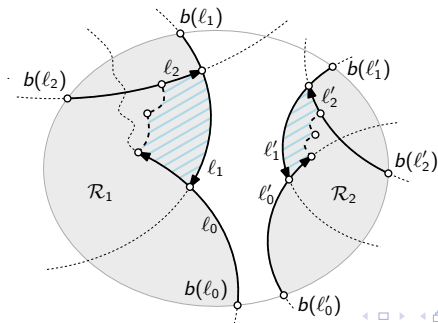
Party-tail of  $\ell$ : part between  $b(\ell)$  and  $\ell \times \pi^{-1}(\ell)$  (or endpoint for  $\ell = \pi(\ell)$ ).

## Lemma

The party-tails of  $\ell$  and  $\ell'$  do not intersect, unless  $\pi(\ell) = \ell'$  or  $\pi(\ell') = \ell$ .

## Lemma

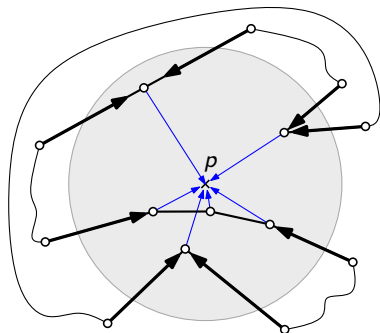
- ▶ Singletons do not exist.
- ▶ There cannot be two parties of size at least three.



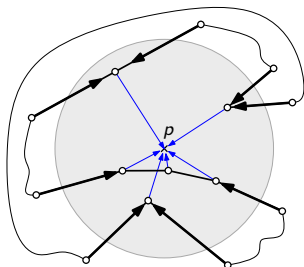
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# Existence of planar matchings

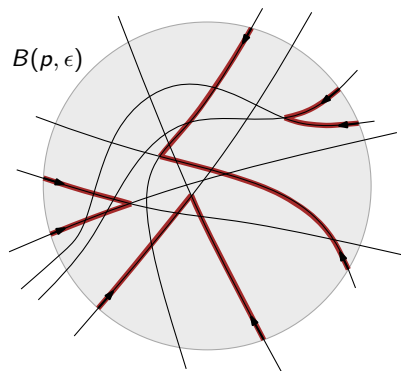
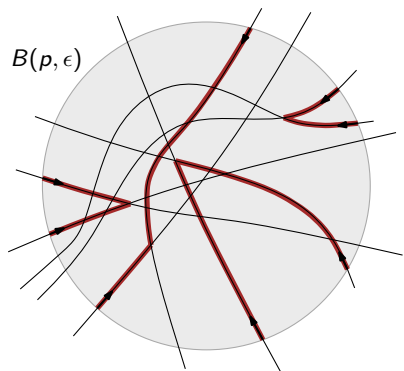


- ▶ Perturb directed edges such that
  - ▶ Edges still reach  $p$  at time  $t$ .
  - ▶ Supporting lines at  $t + \delta$  are in general position.
  - ▶ Perturbed  $\mathcal{W}$  is strictly-planar outside  $B(p, \epsilon)$ .
  - ▶ Vertices do not jump over supporting lines.
- ▶ Compute planar matching.
- ▶ Revert perturbation.

## Lemma

*The new post-event wavefront is weakly-planar.*

# Non-uniqueness



# Summary

Prior status quo:

- ▶ Weighted straight skeletons are half-established:
  - ▶ Algorithms, implementations, theory & practice.
- ▶ Lack of solid foundation.

Contribution:

- ▶ Unified, generalized definition of events.
  - ▶ Maybe interesting for higher dimensions too.
- ▶ First rigorous definition of weighted straight skeletons.
- ▶ We prove that event handling can be always done, i.e., the weighted straight skeleton actually always exists.
- ▶ Planar matchings of directed pseudo-line arrangements might be interesting for their own.

**Acknowledgement:** We thank David Eppstein for suggesting the idea of using stable matchings.

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