Planar Matchings for Weighted Straight Skeletons

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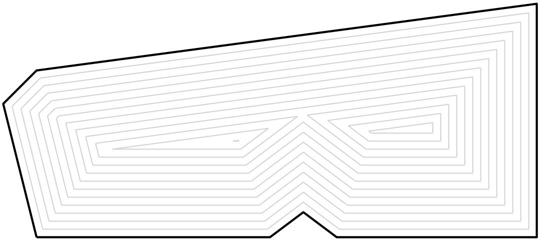
Straight skeletons: A brief introduction



- Introduced by [Aichholzer et al., 1995].

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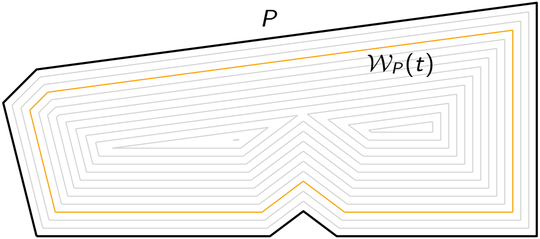

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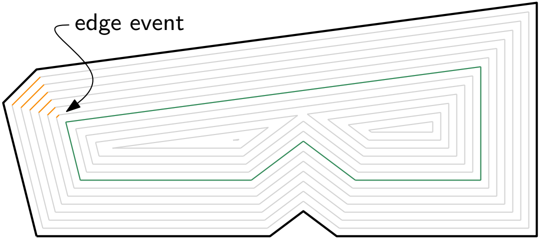

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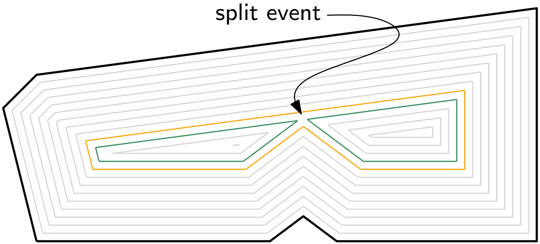

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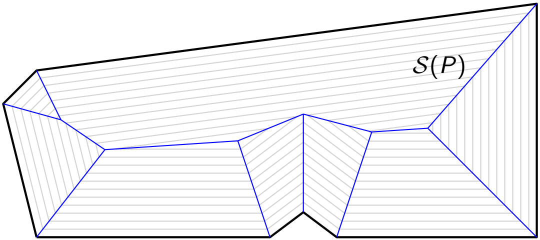

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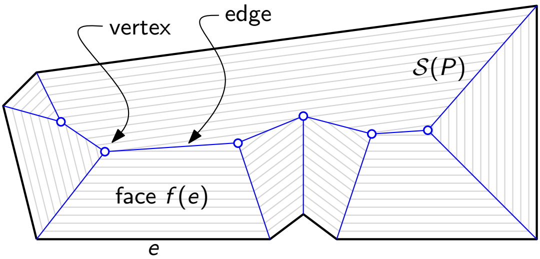

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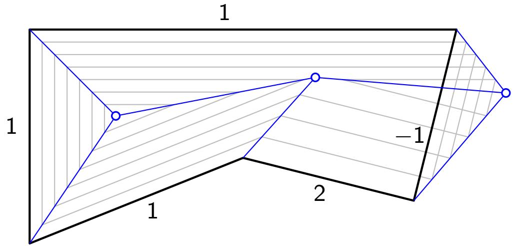
Weighted straight skeletons

- First briefly studied in [Eppstein and Erickson, 1999].
- To every edge e of P a weight $\sigma(e) \neq 0$ is assigned, its speed.



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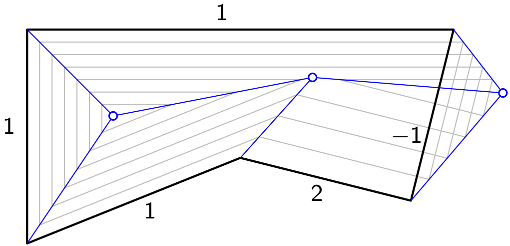

Weighted straight skeletons are "quite established":

- Algorithms were published.
- Implementations are available.
- Used in theory & applications.

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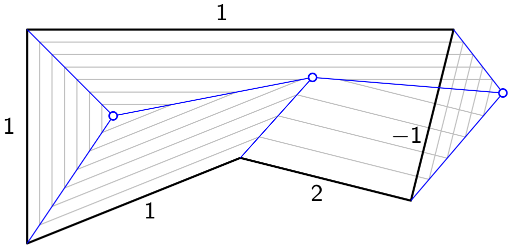

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Still no rigorous definition is known so far!

Prior work

It was silently assumed that weights would not change much.

- Recently we showed that this is not quite true.

Property	Simple polygon			Polygon with holes		
	$\sigma \equiv 1$	σ pos.	σ arb.	$\sigma \equiv 1$	σ pos.	σ arb.
$\mathcal{S}(P)$ is connected	✓	✓	✓	✓	✓	×
$\mathcal{S}(P)$ has no crossing	✓	✓	×	✓	✓	×
$f(e)$ is monotone w.r.t. e	✓	×	×	✓	×	×
bd $f(e)$ is a simple polygon	✓	✓	×	✓	×	×
$\mathcal{T}(P)$ is z-monotone[1]	✓	✓	×	✓	✓	×
$\mathcal{S}(P)$ has $n(\mathcal{S}(P)) - 1 + h$ edges	✓	✓	×	✓	✓	×
$\mathcal{S}(P)$ is a tree	✓	✓	×			

Table: [Biedl et al., 2013, Biedl et al., 2015]

[1] $\mathcal{T}(P) := \bigcup_{t \geq 0} \mathcal{W}_P(t) \times \{t\}$.

Ambiguity of edge events

Ambiguity for parallel edges of different weights become adjacent.

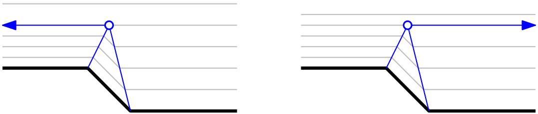

Figure: Resolution methods proposed in [Biedl et al., 2013, Biedl et al., 2015].

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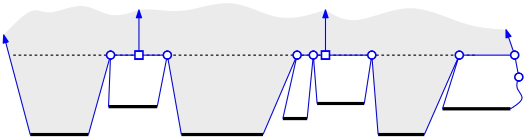
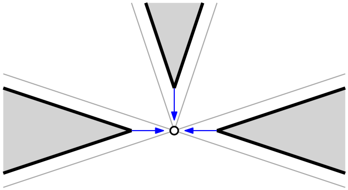


Figure: Resolution methods proposed in [Biedl et al., 2013, Biedl et al., 2015].

Split events: pairing edges

The *standard scheme* works for unweighted straight skeletons.



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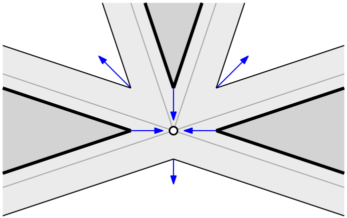
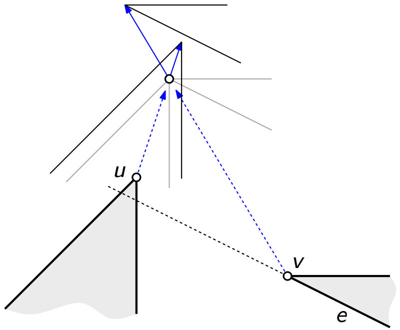


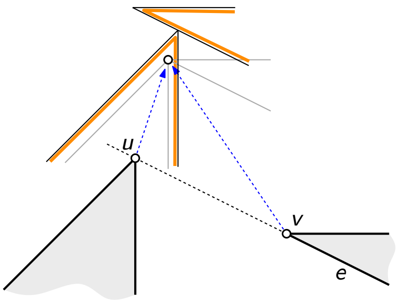
Fundamental principle: Between events, the wavefront is a planar collection of wavefront polygons.

Insufficiency of standard pairing technique



- The standard pairing technique can cause intersections.

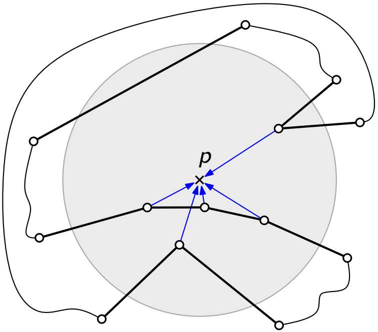
Insufficiency of standard pairing technique



- The standard pairing technique can cause intersections.
- No pairing may exist that gives a planar wavefront.

Split events with many vertices

How to handle this?



Weak planarity

- Let Φ denote the set of all straight-line embeddings $\varphi \colon V \to \mathbb{R}^2$ of a planar graph $G = (V, E)$.
- The pair $(\Phi, d(.,.)_\infty)$ constitutes a metric space, where $d(.,.)_\infty \colon \Phi \times \Phi \to \mathbb{R}$ is defined by $d(\varphi, \varphi')_\infty = \max_{v \in V} \|\varphi(v) - \varphi'(v)\|$.
- Note that the set of planar straight-line embeddings is an open set.

Definition

The set of *weakly-planar* embeddings of G is the topological closure of the set of planar embeddings of G.

Corollary

For every weakly-planar embedding φ and for every $\varepsilon > 0$ there is a planar ε-perturbation φ' of φ, that is, $d(\varphi, \varphi')_\infty < \varepsilon$.

New fundamental principle

At all times, the wavefront is a weakly-planar collection of polygons.

Definition: Event

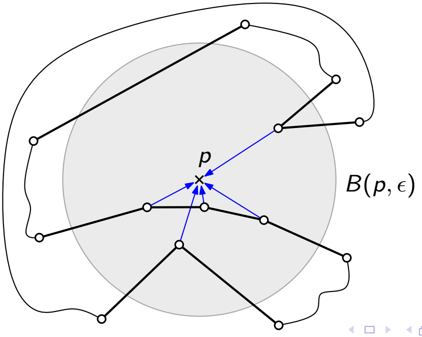
Definition

At location p and time t an *event* happens if at least two vertices meet at time t at p or if $\exists \varepsilon_0 > 0 \; \forall 0 < \varepsilon < \varepsilon_0 \; \exists \delta > 0$ such that

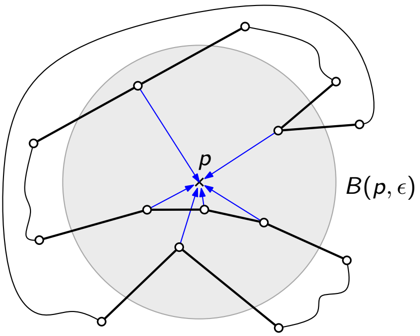
(i) $\mathcal{W}(t') \cap B(p, \varepsilon)$ is non-empty and weakly-planar for $t' \in [t - \delta, t]$ and

(ii) $\mathcal{W}(t') \cap B(p, \varepsilon)$ is non-empty and not weakly-planar for $t' \in (t, t + \delta]$.

We call the edges that meet p at time t the edges that are *involved* in the event.



Pairing edges



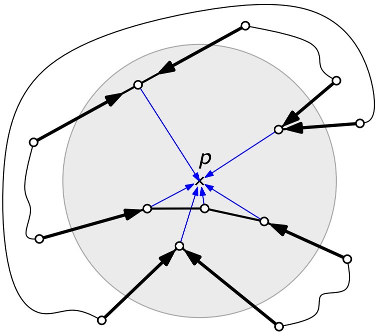
First:

- Remove collapsed edges.
- Split edges where both endpoints are outside $B(p, \varepsilon)$.

Task: Find a pairing of remaining edges to restore weak planarity.

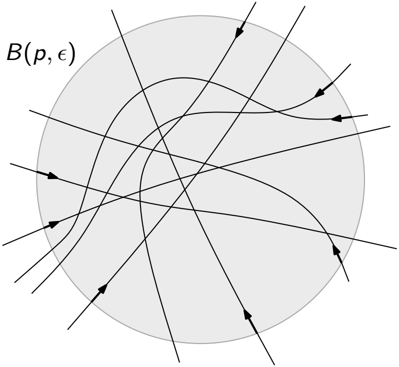
- Is this always possible? Uniquely?

Directed pseudo-line arrangements



- We have k involved chains.
  - Hence, $2k$ (non-collapsed) edges.
  - Assign direction to each edge.
- Consider directed supporting lines of edges, after the event.
  - Perturb a little to obtain general position.
  - $\rightarrow$ pseudo-line arrangement $\mathcal{L}$ of directed pseudo-lines.
  - Obtain a "planar matching" of $\mathcal{L}$ and revert perturbation.

Planar matchings



- Given: even number of directed pseudo lines.
- Every pair intersects, in a single unique point, within $B(p, \epsilon)$.

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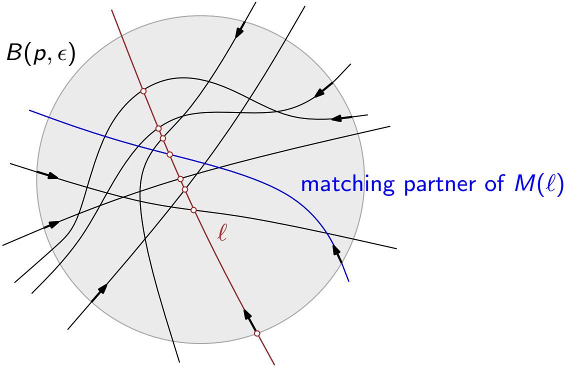

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- Matching: grouping into pairs.

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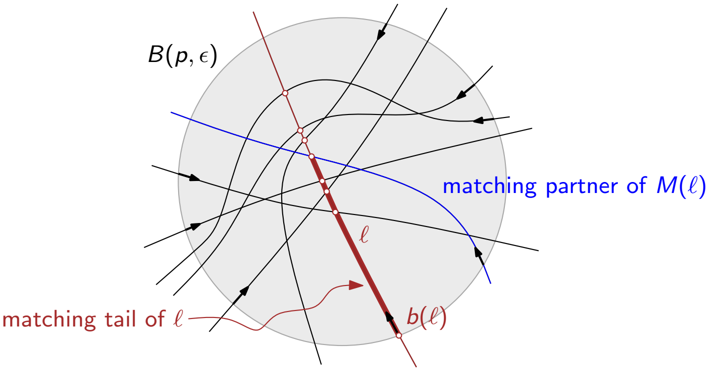

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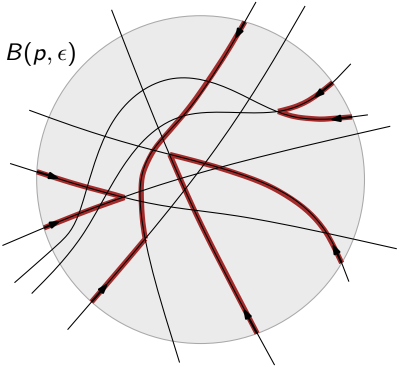
- Planar matching: matching tails do not cross.

Planar matchings



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Planar matchings

Theorem

Every directed pseudo-line arrangement has a planar matching.

Stable roommates

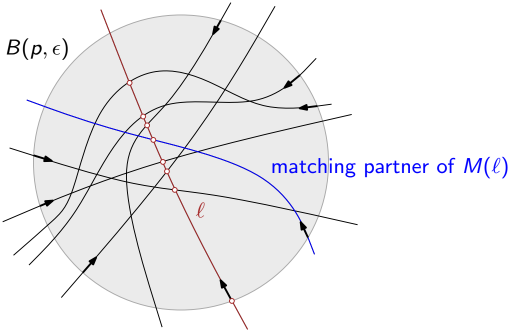
- Given: elements $\mathcal{A} = \{a_1, \ldots, a_N\}$, with N even.
- Each element a_i has a preference list (ranking) of the other elements:
  - Complete: Every element of $\mathcal{A}$ is contained in each ranking, and a_i prefers a_i least.
  - Strict: The ranking is a strict order.
- Consider a matching M on $\mathcal{A}$.
  - Blocking pair $\{a_i, a_j\}$: they prefer each other over their matching partners.
  - M is stable if there is no blocking pair.
- Stable roommates problem: Find a stable matching.
  - Not every instance of the stable roommates problem has a solution!

Stable roommates and planar matchings

- We are given a directed pseudo-line arrangement $\mathcal{L}$.
  - Intersection order gives the ranking.

Lemma

$\mathcal{L}$ has a planar matching if and only if there is a stable matching.



Stable partitions

- A partition is a permutation π of $\ell_1, \ldots, \ell_N$.
  - Decompose π into cycles.
- A cycle of size ≥ 3 is a semi-party if each ℓ in it prefers $\pi(\ell)$ over $\pi^{-1}(\ell)$.
  - In a semi-party partition all cycles of size ≥ 3 are semi-parties.
- Party-blocking pair $\{\ell_i, \ell_j\}$: they prefer each other over $\pi^{-1}(\ell_i)$ and $\pi^{-1}(\ell_j)$.
  - A stable partition is a semi-party partition with no party-blocking pair.

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Theorem ([Tan and Hsueh, 1995])

1. *There is a stable partition, and it can be found in polynomial time.*
2. *There is a stable matching if and only if there is a stable partition with no cycles of odd size.*

Theorem

There are no parties of odd size for directed pseudo-line arrangements.

Odd parties do not exist

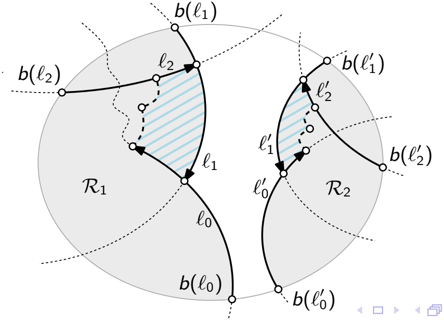
Party-tail of ℓ: part between $b(\ell)$ and $\ell \times \pi^{-1}(\ell)$ (or endpoint for $\ell = \pi(\ell)$).

Lemma

The party-tails of ℓ and ℓ' do not intersect, unless $\pi(\ell) = \ell'$ or $\pi(\ell') = \ell$.

Lemma

- *Singletons do not exist.*
- *There cannot be two parties of size at least three.*



Existence of planar matchings

Theorem

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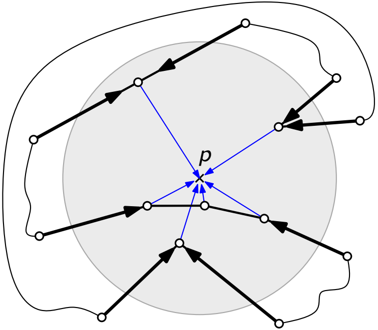

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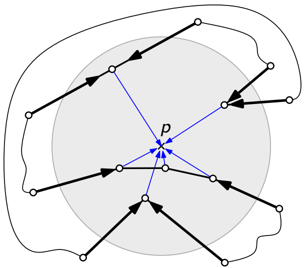
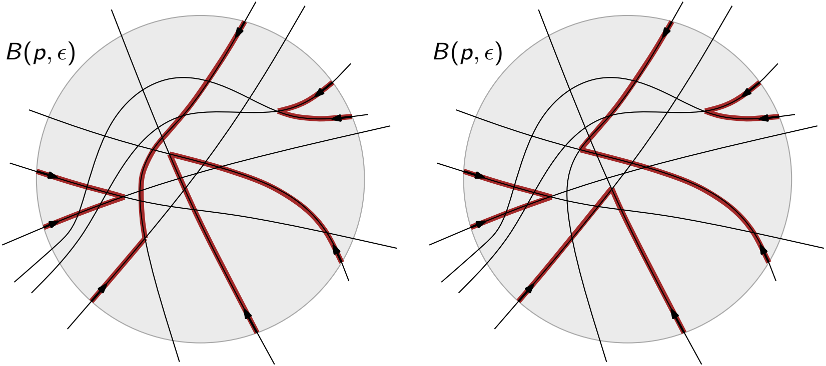


- Perturb directed edges such that
  - Edges still reach p at time t.
  - Supporting lines at $t + \delta$ are in general position.
  - Perturbed $\mathcal{W}$ is strictly-planar outside $B(p, \varepsilon)$.
  - Vertices do not jump over supporting lines.
- Compute planar matching.
- Revert perturbation.

Lemma

The new post-event wavefront is weakly-planar.

Non-uniqueness



Summary

Prior status quo:

- Weighted straight skeletons are half-established:
  - Algorithms, implementations, theory & practice.
- Lack of solid foundation.

Contribution:

- Unified, generalized definition of events.
  - Maybe interesting for higher dimensions too.
- First rigorous definition of weighted straight skeletons.
- We prove that event handling can be always done, i.e., the weighted straight skeleton actually always exists.
- Planar matchings of directed pseudo-line arrangements might be interesting for their own.

Acknowledgement: We thank David Eppstein for suggesting the idea of using stable matchings.

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